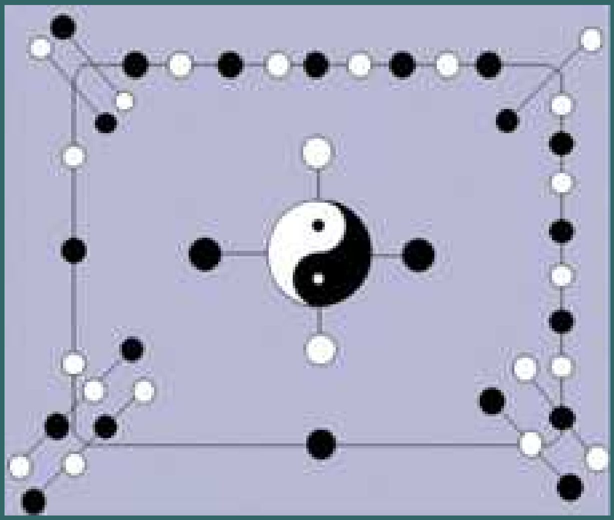
ISBN 978-1-59973-211-4

VOLUME 4, 2012

MATHEMATICAL COMBINATORICS

(INTERNATIONAL BOOK SERIES)

Edited By Linfan MAO



THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

BEIJING UNIVERSITY OF CIVIL ENGINEERING AND ARCHITECTURE

December, 2012

Vol.4, 2012 ISBN 978-1-59973-211-4

Mathematical Combinatorics

(International Book Series)

Edited By Linfan MAO

The Madis of Chinese Academy of Sciences and

Beijing University of Civil Engineering and Architecture

December, 2012

Aims and Scope: The **Mathematical Combinatorics (International Book Series)** (*ISBN 978-1-59973-211-4*) is a fully refereed international book series, quarterly comprising 100-150 pages approx. per volume, which publishes original research papers and survey articles in all aspects of Smarandache multi-spaces, Smarandache geometries, mathematical combinatorics, non-euclidean geometry and topology and their applications to other sciences. Topics in detail to be covered are:

Smarandache multi-spaces with applications to other sciences, such as those of algebraic multi-systems, multi-metric spaces, $\cdots$, etc.. Smarandache geometries;

Differential Geometry; Geometry on manifolds;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Applications of Smarandache multi-spaces to theoretical physics; Applications of Combinatorics to mathematics and theoretical physics;

Mathematical theory on gravitational fields; Mathematical theory on parallel universes;

Other applications of Smarandache multi-space and combinatorics.

Generally, papers on mathematics with its applications not including in above topics are also welcome.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews(USA), Zentralblatt fur Mathematik(Germany), Referativnyi Zhurnal (Russia), Mathematika (Russia), Computing Review (USA), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA).

Subscription A subscription can be ordered by an email to *j.mathematicalcombinatorics@gmail.com* or directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science
Beijing, 100190, P.R.China
Email: maolinfan@163.com

Price: US$48.00

Famous Words:

The mathematics infiltration and control all the theories of natural science branch. It becomes the main measure of scientific achievement marks.

By John Von Neumann, a Hungarian-American mathematician and polymath.

Math.Combin.Book Ser. Vol.4(2012), 01-08

Bicoset of an $(\in v\ q)$-Fuzzy Bigroup

Akinola L.S.[1], Agboola A.A.A.[2] and Oyebo Y.T.[3]

1. Department of Mathematical and computer Sciences, Fountain University, Osogbo, Nigeria

2. Department of Mathematics, University of Agriculture, Abeokuta, Nigeria

3. Department of Mathematics, Lagos State University, Ojoo, Lagos, Nigeria

E-mail: akinolalsb@yahoo.com, aaaola2003@yahoo.com, oyeboyt@yahoo.com

Abstract: In this paper, we define $(\in, \in v\ q)$- fuzzy normal bigroup and $(\in, \in v\ q)$- fuzzy bicoset of a bigroup and discuss their properties as an extension of our work in [2]. We show that an $(\in, \in v\ q)$- fuzzy bigroup of a bigroup G is an $(\in, \in v\ q)$- fuzzy normal bigroup of G if and only if $(\in, \in v\ q)$- fuzzy left bicosets and $(\in, \in v\ q)$- fuzzy right bicosets of G are equal. We also define appropriate algebraic operation on the set of all $(\in, \in v\ q)$- fuzzy normal bigroup of a bigroup G and show that it forms a group.

Key Words: bigroup, fuzzy bigroup, $(\in vq)$-fuzzy bigroup, $(\in, \in vq)$-fuzzy bicosets.

AMS(2010): 03E72, 20D25

§1. Introduction

Zadeh [14]introduced fuzzy set in 1965. Rosenfeld [9] introduced the notion of fuzzy subgroups in 1971. Ming and Ming [8] gave a condition for fuzzy subset of a set to be a fuzzy point, and used the idea to introduce and characterize the notions of quasi coincidence of a fuzzy point with a fuzzy set. Bhakat and Das [3] used these notions by Ming and Ming to introduce and characterize another class of fuzzy subgroup known as $(\in vq)$- fuzzy subgroups. These authors in [4] extended these concepts to $(\in vq)$-fuzzy normal subgroups.

The notion of bigroup was first introduced by P.L.Maggu [5] in 1994. This idea was extended in 1997 by Vasantha and Meiyappan [11]. Meiyappan [7] introduced and characterized fuzzy sub-bigroup of a bigroup in 1998. Akinola and Agboola in [2] introduced the idea of fuzzy singleton to fuzzy bigroup and used it to introduce restricted fuzzy bigroup. These authors also studied the properties of $(\in, \in v\ q)$ fuzzy bigroup.

In this paper, we define $(\in, \in v\ q)$- fuzzy normal bigroup and $(\in, \in v\ q)$- fuzzy bicoset of a bigroup and discuss their properties as an extension of our work in [2]. We show that an $(\in, \in v\ q)$- fuzzy bigroup of a bigroup G is an $(\in, \in v\ q)$- fuzzy normal bigroup of G if and only if $(\in, \in v\ q)$- fuzzy left bicosets and $(\in, \in v\ q)$- fuzzy right bicosets of G are equal. We also show that the set all $(\in, \in v\ q)$- fuzzy normal bigroup of G forms a group under a well defined operation.

[1]Received July 12, 2012. Accepted November 15, 2012.

§2. Preliminary Results

Definition 2.1([5,6]) *A set $(G, +, \cdot)$ with two binary operations $+$ and $\cdot$ is called a bi-group if there exist two proper subsets G_1 and G_2 of G such that*

(i) $G = G_1 \cup G_2$;
(ii) $(G_1, +)$ *is a group;*
(iii) $(G_2, \cdot)$ *is a group.*

Definition 2.2)([5]) *A subset $H(\neq 0)$ of a bi-group $(G, +, \cdot)$ is called a sub bi-group of G if H itself is a bi-group under the operations of $+$ and $\cdot$ defined on G.*

Theorem 2.3([5]) *Let $(G, +, \cdot)$ be a bigroup. If the subset $H \neq 0$ of a bigroup G is a sub bigroup of G, then $(H, +)$ and $(H, \cdot)$ are generally not groups.*

Definition 2.4([12]) *Let G be a non empty set. A mapping $\mu : G \to [0, 1]$ is called a fuzzy subset of G.*

Definition 2.5([12]) *Let μ be a fuzzy set in a set G. Then, the level subset μ_t is defined as $\mu_t = \{x \in G : \mu(x) \geq t\}$ for $t \in [0, 1]$.*

Definition 2.6([9]) *Let μ be a fuzzy set in a group G. Then, μ is said to be a fuzzy subgroup of G, if the following hold:*

(i) $\mu(xy) \geq \min\{\mu(x), \mu(y)\} \ \ \forall\, x, y \in G$;
(ii) $\mu(x^{-1}) = \mu(x) \ \forall\, x \in G$.

Definition 2.7([11]) *Let μ_1 be a fuzzy subset of a set X_1 and μ_2 be a fuzzy subset of a set X_2, then the fuzzy union of the sets μ_1 and μ_2 is defined as a function $\mu_1 \cup \mu_2 : X_1 \cup X_2 \longrightarrow [0, 1]$ given by:*

$$(\mu_1 \cup \mu_2)(x) = \begin{cases} max(\mu_1(x), \mu_2(x)) & if\ x \in X_1 \bigcap X_2, \\ \mu_1(x) & if\ x \in X_1\ \&\ x \notin X_2, \\ \mu_2(x) & if\ x \in X_2\ \&\ x \notin X_1. \end{cases}$$

Definition 2.8([2]) *Let $G = G_1 \cup G_2$ be a bi-group. Let $\mu = \mu_1 \cup \mu_2$ be a fuzzy bigroup. A fuzzy subset $\mu = \mu_1 \cup \mu_2$ of the form:*

$$\mu(x) = \begin{cases} M(t, s) \neq 0 & if\ x = y \in G, \\ 0 & if\ x \neq y. \end{cases}$$

where $t, s \in [0, 1]$ such that

$$\mu_1(x) = \begin{cases} t \neq 0 & if\ x = y \in G_1, \\ 0 & if\ x \neq y. \end{cases}$$

and

$$\mu_2(x) = \begin{cases} s \neq 0 & if\ x = y\ \in G_2, \\ 0 & if\ x \neq y. \end{cases}$$

is said to be a fuzzy point of the bi-group G with support x and value $M(t,s)$, and denoted by $x_{M(t,s)}$.

Definition 2.9([2]) *A fuzzy point $x_{M(t,s)}$ of the bigroup $G = G_1 \cup G_2$, is said to belong to (resp. be quasi coincident with) a fuzzy subset $\mu = \mu_1 \cup \mu_2$ of G, written as $x_{M(t,s)} \in \mu$[resp. $x_{M(t,s)} q\mu$] if $\mu(x) \geq M(t,s)$(resp. $\mu(x) + M(t,s) > 1$). $x_{M(t,s)} \in \mu$ or $x_{M(t,s)}\ q\mu$ will be denoted by $x_M(t,s) \in vq\mu$.*

Definition 2.10([2]) *A fuzzy bisubset μ of a bigroup G is said to be an ($\in v\ q$)- fuzzy sub bigroup of G if for every $x,\ y\ \in\ G$ and $t_1,\ t_2,\ s_1,\ s_2,\ t,\ s,\ \in\ [0,1]$,*

(*i*) $x_{M(t_1,t_2)} \in \mu,\ y_{M(s_1,s_2)} \in \mu\ \Rightarrow\ (xy)_{M(t,s)} \in vq\mu$;
(*ii*) $x_{M(t_1,t_2)} \in \mu\ \Rightarrow\ (x^{-1})_{M(t_1,t_2)} \in vq\mu$, *where* $t = M(t_1,t_2)$ *and* $s = M(s_1,s_2)$.

Theorem 2.11([2]) *Let $\mu = \mu_1 \cup \mu_2 :\ G = G_1 \cup G_2 \rightarrow [0,1]$ be a fuzzy subset of G. Suppose that μ_1 is an ($\in vq$)-fuzzy subgroup of G_1 and μ_2 is an ($\in vq$)-fuzzy subgroup of G_2, then μ is an ($\in vq$)-fuzzy subgroup of G.*

§3. Main Results

Definition 3.1 *An ($\in v\ q$)- fuzzy bigroup μ of a bigroup G is said to be an ($\in v\ q$)- fuzzy normal bigroup of G if for any $x,\ y\ \in\ G$ and $t_1,\ t_2,\ \in\ [0,1]$, $x_{M(t_1,t_2)} \in \mu\ \Rightarrow\ (yxy^{-1})_{M(t_1,t_2)} \in vq\mu$.*

Theorem 3.2 *Let $\mu = \mu_1 \cup \mu_2 :\ G = G_1 \cup G_2 \rightarrow [0,1]$ be a fuzzy bi-subset of G. Suppose that μ_1 is an ($\in vq$)-fuzzy normal subgroup of G_1 and μ_2 is an ($\in vq$)-fuzzy normal subgroup of G_2, then μ is an ($\in vq$)-fuzzy normal bi-group of G.*

Proof That μ is an ($\in vq$)-fuzzy bigroup is clear from Theorem 2.11. To now show that it is normal. Suppose that μ_1 and μ_2 are ($\in vq$)-fuzzy normal subgroups of G_1 and G_2 respectively. For $x,\ y\ \in\ G$ and $t_1,\ t_2,\ \in\ [0,1]$,

$$x_{t_1} \in \mu_1\ \Rightarrow\ (yxy^{-1})_{t_1} \in vq\mu_1$$

and

$$x_{t_2} \in \mu_2\ \Rightarrow\ (yxy^{-1})_{t_2} \in vq\mu_2.$$

So that

$$\mu_1(yxy^{-1}) \geq t_1\ or\ \mu_1(yxy^{-1}) + t_1 > 1$$

and

$$\mu_2(yxy^{-1}) \geq t_2\ or\ \mu_2(yxy^{-1}) + t_2 > 1,$$

which shows that

$$max\{\mu_1(yxy^{-1}),\ \mu_2(yxy^{-1})\} \geq M(t_1,t_2)\ or\ max\{\mu_1(yxy^{-1}),\ \mu_2(yxy^{-1})\} + M(t_1,t_2) > 1.$$

Thus

$$\mu_1 \cup \mu_2(yxy^{-1}) \geq M(t_1,t_2)\ or\ \mu_1 \cup \mu_2(yxy^{-1}) + M(t_1,t_2) > 1.$$

So

$$\mu(yxy^{-1}) \geq M[t_1,t_2]\ or\ \mu(yxy^{-1}) + M[t_1,t_2]\ >\ 1,$$

which concludes that $(yxy^{-1})_{M(t_1,t_2)}\ \in vq\mu$. □

Definition 3.3 *Let* $\mu = \mu_1 \cup \mu_2 :\ G = G_1 \cup G_2\ \to [0,1]$ *be a fuzzy bigroup of a bigroup* G. *For* $x \in G$, μ^l_x *(res* μ^r_x*)* $:\ G \to [0,1]$ *defined as*

$$\mu^l_x(g) = \mu(gx^{-1})\ (res\ \mu^r_x(g) = \mu(x^{-1}g)$$

is called an $(\in,\ \in)$*- fuzzy left(resp. fuzzy right) cosets of* G *determined by* x *and* μ.

Remark 3.4 Let μ be a fuzzy bigroup of a bigroup G, then μ is an $(\in,\ \in)$- fuzzy normal bigroup of a G if and only if $\mu^l_x(g)\ =\ \mu^r_x(g)$.

$(\in,\ \in)$- fuzzy bigroup here refers to fuzzy bigroup that satisfy Meiyappian's fuzzy bigroup conditions.

Example 3.5 Let $G = \{e,\ a,\ b,\ c,\ d,\ f,\ x,\ y,\ z,\ w\}$ be a bigroup where $G_1 = \{e,\ a,\ b,\ c,\ d,\ f,\}$ with the cayley table

×	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	c	f	e	d
b	b	e	a	d	f	c
c	c	d	f	e	a	b
d	d	f	c	b	e	a
f	f	c	d	a	b	e

and $G_2 = \{x,\ y,\ z,\ w,\}$ with Cayley table given below

∘	x	y	z	w
x	x	y	z	w
y	y	x	w	z
z	z	w	x	y
w	w	z	y	x

be the constituting subgroups. Define $\mu = \mu_1 \cup \mu_2 : G = G_1 \cup G_2 \to [0,1]$ as $\{0.6,\ 0.75,\ 0.8,\ 0.4,\ 0.4,\ 0.4\}$ for $\{e,\ a,\ b,\ c,\ d,\ f,\}$ respectively, and $\{0.6,\ 0.3,\ 0.3,\ 0.5\}$ for $\{x,\ y,\ z,\ w,\}$ respectively. It is also easy to see that fuzzy bisubset μ so defined on the bigroup G is an $(\in, \in vq)$ fuzzy bigroup. Now consider

$$0.6 = \mu(e) = \mu_1(e) = \mu_1(bb^{-1}) = \mu_1(ba)$$
$$\leq \min\{\mu_1(b), \mu_1(a)\}$$
$$= \min\{0.75, 0.7\} = 0.7$$

even though

$$0.6 = \mu_2(x) = \mu_2(zz^{-1}) = \mu_2(zz)$$
$$\geq max\{\mu_2(z), \mu_2(z)\} = 0.3.$$

Hence, μ so defined on the bigroup G is an $(\in, \in)$ fuzzy bigroup.

Also, consider

$$\mu^l_d(c) = \mu^l_{1d}(c) = \mu_1(cd^{-1}) = \mu_1(cd) = \mu_1(a) = 0.75$$

since $\mu^l_{2d}(c) = 0$ and

$$\mu^r_d(c) = \mu^r_{1d}(c) = \mu_1(d^{-1}c) = \mu_1(dc) = \mu_1(b) = 0.8$$

also since $\mu^r_{2d}(c) = 0$. Even though,

$$\mu^l_y(z) = \mu^l_{2y}(z) = \mu_2(yz^{-1}) = \mu_2(yz) = \mu_2(w) = 0.5$$

and

$$\mu^r_y(z) = \mu^r_{2y}(z) = \mu_2(z^{-1}y) = \mu_2(zy) = \mu_2(w) = 0.5$$

where $\mu^l_{1r}(z) = 0$ and $\mu^r_{1r}(z) = 0$. It is clear that $\mu^l_x \neq \mu^r_x$ generally.

Definition 3.6 *Let μ be a fuzzy bigroup of a bigroup G. For any $x \in G$, $\hat{\mu}_x$(resp.$\check{\mu}_x$) $: G \to [0,1]$ defined by*

$$\hat{\mu}_x(g) = M[\mu(gx^{-1}), 0.5] \ (resp.\ \check{\mu}_x(g) = M[\mu(gx^{-1}), 0.5]$$

for every $g \in G$ is called $(\in, \in vq)$-fuzzy left bicoset (resp.$(\in, \in vq)$-fuzzy right bicoset) of G determined by x and μ.

Theorem 3.7 *Let μ be a fuzzy bigroup of a bigroup G. Then μ is an $(\in, \in vq)$-fuzzy normal bigroup of G if*

$$\hat{\mu}_x = \check{\mu}_x \ \forall\ x \in G.$$

Proof Let μ be an $(\in, \in vq)$-fuzzy normal bigroup of G. Let $x \in G$, then $\forall\ g \in G$, if $x,\ g \in G \setminus G_2$,

$$\begin{aligned} \hat{\mu}_x(g) &= (\hat{\mu_1} \cup \hat{\mu_2})_x(g) = \hat{\mu}_{1x}(g) = M[\mu_1(gx^{-1}), 0.5] \\ &\geq M[\mu_1(x^{-1}g), 0.5] = \check{\mu}_{1x}(g) = \check{\mu}_x(g) \end{aligned}$$

Therefore, $\hat{\mu}_x(g) \geq \check{\mu}_x(g)$. By similar argument, we can show that $\check{\mu}_x(g) \geq \hat{\mu}_x(g)$ for all $x,\ g \in G \setminus G_2$.

If $x,\ g \in G \setminus G_1$, then,

$$\hat{\mu}_x(g) = (\hat{\mu_1} \cup \hat{\mu_2})_x(g) = \hat{\mu}_{2x}(g) = M[\mu_2(gx^{-1)}, 0.5]$$

$$\geq M[\mu_2(x^{-1}g), 0.5] = \check{\mu}_{2x}(g) = \check{\mu}_x(g)$$

Therefore, $\hat{\mu}_x(g) \geq \check{\mu}_x(g)$. By similar argument, we can show that $\check{\mu}_x(g) \geq \hat{\mu}_x(g)$ for all $x,\ g \in G \setminus G_1$.

If $x,\ g \in G_1 \cap G_2$, then,

$$\begin{aligned} \hat{\mu}_x(g) &= (\hat{\mu_1} \cup \hat{\mu_2})_x(g)) = \max\{\hat{\mu_1}(g), \hat{\mu_2}(g) \\ &= \max\{M[\mu_1(gx^{-1}), 0.5], M[\mu_2(gx^{-1}), 0.5]\} \\ &\geq \max\{M[\mu_1(x^{-1}g), 0.5], M[\mu_2(x^{-1}g), 0.5]\} \\ &= \max\{\check{\mu}_{1x}(g), \check{\mu}_{2x}(g) = (\check{\mu}_1 \cup \check{\mu}_2)_x(g) = \check{\mu}_x(g). \end{aligned}$$

Therefore, $\hat{\mu}_x(g) \geq \check{\mu}_x(g)$ for all $x,\ g \in G_1 \cap G_2$. Similar argument shows that $\check{\mu}_x(g) \geq \hat{\mu}_x(g)$. Hence, $\hat{\mu}_x = \check{\mu}_x \ \forall\ x \in G$. □

Theorem 3.8 *Let μ be a fuzzy bigroup of a bigroup G. If $\hat{\mu}_x = \check{\mu}_x \ \forall\ x \in G$, then μ is an $(\in, \in vq)$-fuzzy normal bigroup of G.*

Proof The theorem is a direct converse of Theorem 3.1.6. Let $\hat{\mu}_x = \check{\mu}_x \ \forall\ x \in G$, then, for all $g \in G$,

$$(\hat{\mu_1} \cup \hat{\mu_2})_x(g) = (\check{\mu}_1 \cup \check{\mu}_2)_x(g)$$

which implies that

$$\max\{M[\mu_1(gx^{-1}), 0.5], M[\mu_2(gx^{-1}), 0.5]\} = \max\{M[\mu_1(x^{-1}g), 0.5], M[\mu_2(x^{-1}g), 0.5]\}.$$

if we replace g by xyx, it follows that

$$\max\{M[\mu_1(xy), 0.5], M[\mu_2(xy), 0.5]\} = \max\{M[\mu_1(yx), 0.5], M[\mu_2(yx), 0.5]\},$$

which shows that μ, which is a fuzzy bigroup of the bigroup G is normal. That μ is an $(\in, \in vq)$-fuzzy normal bigroup of G is a direct consequence of equivalent conditions of Proposition 2.4.1. Hence the proof. □

What can we say about the properties of a set that contains all the $(\in, \in vq)$- fuzzy normal bigroup of a bigroup G? Can an appropriate operation be defined on this set to form a group or a normal subgroup of that set? The following observations have been made to give an insight into the answers:

In a bigroup $G = G_1 \cup G_2$, if we let μ_1 be a normal fuzzy subgroup of G_1 and S, the set of all fuzzy cosets $\hat{\mu}_1$ of μ_1 in G_1. If we follow the approach used for similar concept in [4], define composition on S as:

$$\hat{\mu}_{1x} \cdot \hat{\mu}_{1y} = \hat{\mu}_{1xy} \qquad \forall\ \ x,\ y\ \in\ G.$$

For any $g \in G_1$, if we let

$$\hat{\mu}_{1x}(g) = \hat{\mu}_{1y}(g) \;\; and \;\; \hat{\mu}_{1z}(g) = \hat{\mu}_{1w}(g),$$

then

$$M[\mu_1(gx^{-1}), 0.5] = M[\mu_1(gy^{-1}), 0.5] \tag{$\star$}$$

and

$$M[\mu_1(gz^{-1}), 0.5] = M[\mu_1(gw^{-1}), 0.5], \tag{$\star\star$}$$

so that

$$\hat{\mu}_{1xz}(g) = M[\mu_1(gz^{-1}x^{-1}), 0.5] = M[\mu_1(gz^{-1}x^{-1}), 0.5].$$

By replacing g by gz^{-1} in ($\star$).

$$M[\mu_1(gz^{-1}x^{-1}), 0.5] \geq M\{M[\mu_1(y^{-1}gz^{-1}),\; 0.5\;], 0.5\}$$

and since μ_1 is fuzzy normal, it follows that

$$M\{M[\mu_1(y^{-1}gz^{-1}),\; 0.5\;], 0.5\} \geq M[\mu_1(y^{-1}gw^{-1}),\; 0.5]$$

replacing g by $y^{-1}g$ in ($\star\star$).

$$M[\mu_1(y^{-1}gw^{-1}),\; 0.5] \geq M[\mu_1(gw^{-1}y^{-1}),\; 0.5]$$

and since μ_1 is fuzzy normal, it follows that

$$M[\mu_1(gw^{-1}y^{-1}),\; 0.5] \geq \hat{\mu}_{1yw}(g).$$

By a similar argument, it can be shown that $\hat{\mu}_{1yw}(g) \geq \hat{\mu}_{1xz}(g) \;\; \forall \;\; g \in G$, so that $\hat{\mu}_{1xz} = \hat{\mu}_{1yw}$, which shows that the composition defined on S is well defined.

It is easy to see that S is a group with the identity element $\hat{\mu}_{1e}$, and $\hat{\mu}_{1x^{-1}}$ as the inverse of $\hat{\mu}_{1x}$ for every $x \in G_1$. Let $\bar{\mu} : S \to [0, 1]$ be defined by

$$\bar{\mu}(\hat{\mu}_{1x}) = \mu_1(x) \qquad\quad \forall x \in G_1,$$

it is observed that

$$\begin{aligned}\bar{\mu}(\hat{\mu}_{1x} \cdot \hat{\mu}_{1y^{-1}}) &= \mu(\hat{\mu}_{1xy^{-1}}) = \mu_1(xy^{-1}) \\ &= M[\mu_1(x),\; \mu_1(y),\;\; 0.5] \\ &= M[\bar{\mu}(\hat{\mu}_{1x}),\; \bar{\mu}(\hat{\mu}_{1y}),\;\; 0.5] \;\;\; \forall \;\;\; \hat{\mu}_{1x},\; \hat{\mu}_{1y} \;\in\; S.\end{aligned}$$

Also,

$$\begin{aligned}\bar{\mu}(\hat{\mu}_{1x}\hat{\mu}_{1a}\hat{\mu}_{1x^{-1}}) &= \bar{\mu}(\hat{\mu}_{1xax^{-1}}) = \mu_1(xax^{-1}) \\ &= M[\mu_1(a),\;\; 0.5]\end{aligned}$$

since μ is fuzzy normal,

$$M[\mu_1(a),\;\; 0.5] = M[\bar{\mu}(\hat{\mu}_{1x},\;\; 0.5].$$

which shows that $\bar{\mu}$ is a fuzzy normal subgroup of S.

Now that it has been established that in a bigroup $G = G_1 \cup G_2$, if μ_1 is a normal fuzzy subgroup of G_1 and S_1, the set of all fuzzy cosets $\hat{\mu}_1$ of μ_1 in G_1, is a normal subgroup with respect to a well defined operation.

By extended implication, we can say that in a bigroup $G = G_1 \cup G_2$, if μ_2 is a normal fuzzy subgroup of G_2 and S_2, the set of all fuzzy cosets $\hat{\mu}_2$ of μ_2 in G_2, is a normal subgroup with respect to a well defined operation, so that we can then conclude that in a bigroup $G = G_1 \cup G_2$, if $\mu = \mu_1 \cup \mu_2$ is a normal fuzzy subgroup of G and S, the set of all fuzzy cosets $\hat{\mu}$ of μ in G is a normal subgroup with respect to a well defined operations S.

This result is summarized below:

Corollary 3.9 *Let $G = G_1 \cup G_2$ be a bigroup. If $\mu = \mu_1 \cup \mu_2$ is a normal fuzzy subgroup of G, the set S, of all fuzzy cosets $\hat{\mu}$ of μ in G is a normal subgroup with respect to a well defined operations on S.*

References

[1] Ajmal N., Thomas K.V., Quasinormality and fuzzy subgroups, *Fuzzy Sets and Systems*, 58(1993) 217-225.

[2] Akinola L. S. and Agboola A.A.A., On ($\in vq$)-Fuzzy Bigroup, *International Journal of Mathematical Combinatorics*, Vol.4 (2010), 01-07.

[3] Bhakat S. K., Das P., On the definition of fuzzy Subgroups, *Fuzzy Sets and Systems*, 51(1992), 235-241.

[4] Bhakat S. K., Das P., ($\in, \in vq$)-fuzzy subgroup, *Fuzzy Sets and Systems*, 80(1996), 359-393.

[5] Maggu P. L., On introduction of bigroup concept with its application in industry, *Pure Appl. Math Sci.*, 39, 171-173(1994).

[6] Maggu P.L. and Rajeev K., On sub-bigroup and its applications, *Pure Appl. Math Sci.*,43, 85-88(1996).

[7] Meiyappan D., *Studies on Fuzzy Subgroups,* Ph.D. Thesis, IIT(Madras), June 1998.

[8] Ming P.P., Ming L.Y., Fuzzy topology I: Neighbouhood structure of a fuzzy point and Moore-Smith convergence, *J. Math. Anal.Appl.*, 76(1980) 571-599.

[9] Rosenfeld A., Fuzzy groups, *J. Math. Anal.Appl.*, 35(1971) 512-517.

[10] Vasantha Kandasamy W. B., *Bialgebraic Structures and Smarandache Bialgebraic Structures,* American Research Press, Rehoboth, NM, 2003.

[11] Vasantha Kandasamy W. B. and Meiyappan D., *Bigroup and fuzzy bigroup,* 63*rd* Annual Conference, Indian Mathematical Society, December, 1997.

[12] Zadeh, L. A., Fuzzy sets,*Inform. and Control*, 8(1965), 338-353.

Math.Combin.Book Ser. Vol.4(2012), 09-16

The Characterizations of Nonnull Inclined Curves in Lorentzian Space $\mathbf{L}^5$

Handan Balgetir Öztekin and Serpil Tatlıpınar

Department of Mathematics, Firat University, 23119 Elazig, TÜRKİYE

E-mail: handanoztekin@gmail.com, s.tatlipinar@hotmail.com

Abstract: In this paper, making use of the author's method appeared in [1] , we define nonnull inclined curve in L^5. We also give some new characterizations of these curves in Lorentzian 5-space L^5.

Key Words: Inclined curves, Lorentzian 5-space, Frenet frame.

AMS(2010): 53A04, 53B30, 53C40

§1. Introduction

A helix is a curve, the tangent of which makes a constant angle with a fixed line. Standard screws, bolts and a double-stranded molecule of DNA are the most common examples for helices in the nature and structures. First, Lancret in 1802 gave the characterizations of this curve. He obtained that "a curve is a helix if and only if ratio of curvature k_1 to torsion k_2 is costant". In [5], Ç. Camcı, K. İlarslan, L. Kula and H. Hilmi Hacısalihoğlu studied generalized helices in E^n and N.Ekmekci, H. Hilmi Hacısalihoğlu investigated harmonic curvatures in Lorentzian space in [6]. Then A. Altın obtained helix in R^n_v in [4]. Recently, in [1,2,3] , T. A. Ali studied inclined curves and slant helices in E^5 and E^n.

In this study, by considering inclined curves in the Euclidean 5-space E^5 as given in [1], we investigate necessary and sufficient conditions to be inclined for a nonnull curve in Lorentzian 5-space L^5 and obtain some characterizations of nonnull inclined curve in terms of their curvatures.

§2. Preliminaries

Let $\alpha : I \subset R \to L^5$ be a regular curve in L^5. The curve α is spacelike if all of its velocity vectors are spacelike, and timelike and null can be defined similarly. If $\langle \alpha'(t), \alpha'(t)\rangle = \pm 1$, then α is called a unit speed curve, where $\langle , \rangle$ detones the scalar product of L^5.

Let $\alpha : I \subset R \to L^5$ be a regular curve in L^5 and $\psi = \{\alpha'(t), \alpha''(t), \alpha'''(t), \alpha^{iv}(t), \alpha^{v}(t)\}$ a maximal linear independent and nonnull set. An orthonormal system $\{V_1(t), V_2(t), V_3(t), V_4(t),$

[1]Received October 30, 2012. Accepted November 25, 2012.

$V_5(t)\}$ can be obtained from ψ. This is called a Serre-Frenet frame at the point $\alpha(t)$.

Defiinition 2.1([4]) *Let α be a unit speed curve in L^5 and let the set $\{V_1(t), V_2(t), V_3(t), V_4(t),$ $V_5(t)\}$ be the Serre-Frenet frame at the point $\alpha(t)$. Then, the following hold*

$$\begin{aligned}
&V_1'(t) = \epsilon_1(t)k_1(t)V_2(t), \\
&\vdots \\
&V_i'(t) = -\epsilon_i(t)k_{i-1}(t)V_{i-1}(t) + \epsilon_i(t)k_i(t)V_{i+1}(t),\ 1 < i < 5 \qquad (2.1) \\
&\vdots \\
&V_5'(t) = -\epsilon_5(t)k_4(t)V_4(t),
\end{aligned}$$

where $\epsilon_i(t)$ denotes $\langle V_i(t), V_i(t)\rangle = \pm 1,$.

Definition 2.2([1]) *A unit speed curve $\alpha : I \to E^5$ is is said to be an inclined curve if its tangent T makes a constant angle with a fixed direction U.*

Theorem 2.1([1]) *Let $\alpha : I \to E^5$ be a unit speed curve regular curve in E^5. Then α is an inclined curve if and only if the function*

$$\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left[\left(\frac{k_1}{k_2}\right)'\right]^2 + \frac{1}{k_4^2}\left[\frac{k_1k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]^2$$

is a constant. Moreover, this constant agrees with $\tan^2\theta$, being θ the angle that makes T with the fixed direction U that determines α.

§3. Characterizations of Nonnull Inclined Curves in $\mathbf{L^5}$

In this section, we define nonnull inclined curves in L^5. We also give some new characterizations of these curves in Lorentzian 5-space L^5.

Definition 3.1 *A unit speed nonnull curve $\gamma : I \to L^5$ is called a nonnull inclined curve in L^5 if its first Frenet vector V_1 makes a constant angle with a fixed direction U.*

Theorem 3.1 *Let $\gamma : I \to L^5$ be a unit speed nonnull curve in Lorentzian space L^5. Then γ is a nonnull inclined curve if and only if the function*

$$\frac{\epsilon_5}{\epsilon_3}\sqrt{\epsilon_4\epsilon_5}\left(\frac{k_1}{k_2}\right)^2 + \left[\frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{1}{\epsilon_4k_4}\left[\frac{\epsilon_4k_1k_3}{k_2} + \left[\frac{1}{\epsilon_3k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]^2$$

is constant.

Proof Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in L^5. Assume that γ is a nonnull inclined curve. Let U be the fixed direction which makes a costant angle ϕ with $V_{1.}$ Consider the differentiable functions $a_{i,}$ $1 \le i \le 5$,

$$U = \sum_{i=1}^{5} a_i(s)V_i(s), \quad s \in I, \qquad (3.1)$$

that is

$$a_i =< V_i, U >, \quad 1 \leq i \leq 5.$$

Then the function $a_1(s) =< V_1(s), U >$ is constant, that is

$$a_1(s) =< V_1(s), U >= \begin{cases} \cos\phi = const, & \text{if } \gamma \text{ is a spacelike curve} \\ \cos h\phi = const, & \text{if } \gamma \text{ is a timelike curve} \end{cases} \tag{3.2}$$

By differentiation of (3.2) with respect to s and using the Frenet formula (2.1), we have

$$a_1'(s) = -\epsilon_1 k_1 a_2 = 0. \tag{3.3}$$

Then $a_2 = 0$ and therefore U is in the subspace $Sp\{V_1, V_3, V_4, V_5\}$. Because the vector field U is constant, a differentiation in (3.1) together (2.1) gives the following system of ordinary differential equation:

$$\begin{aligned} &\epsilon_2 k_1 a_1 - \epsilon_2 k_2 a_3 = 0, \\ &a_3' - \epsilon_3 k_3 a_4 = 0, \\ &a_4' + \epsilon_4 k_3 a_3 - \epsilon_4 k_4 a_5 = 0, \\ &a_5' + \epsilon_5 k_4 a_4 = 0. \end{aligned} \tag{3.4}$$

The first three equations in (3.4) lead to

$$\begin{aligned} a_3 &= \frac{k_1}{k_2} a_1, \\ a_4 &= \frac{1}{\epsilon_3 k_3} \left(\frac{k_1}{k_2}\right)' a_1, \\ a_5' &= \frac{1}{\epsilon_4 k_4} \left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3} \left(\frac{k_1}{k_2}\right)'\right]'\right] a_1 \end{aligned} \tag{3.5}$$

We do the change of variables:

$$t(s) = \int^s k_4(u) du \quad \frac{dt}{ds} = k_4(s).$$

In particular, and from equation (3.4), we have

$$a_4'(t) = \epsilon_4 a_5(t) - \epsilon_4 \left(\frac{k_3(t)}{k_4(t)}\right) a_3(t).$$

As the last equation of (3.4) yields

$$a_5''(t) + \epsilon_4 \epsilon_5 a_5(t) = \epsilon_4 \epsilon_5 \left(\frac{k_1(t) k_3(t)}{k_2(t) k_4(t)}\right) a_1. \tag{3.6}$$

The general solution of this equation is obtained

$$\begin{aligned} a_5(t) = [&\left(A - \int \frac{k_1(t) k_3(t)}{k_2(t) k_4(t)} \sin\sqrt{\epsilon_4 \epsilon_5} t dt\right) \cos\sqrt{\epsilon_4 \epsilon_5} t \\ &+ \left(B + \int \frac{k_1(t) k_3(t)}{k_2(t) k_4(t)} \cos\sqrt{\epsilon_4 \epsilon_5} t dt\right) \sin\sqrt{\epsilon_4 \epsilon_5} t] \sqrt{\epsilon_4 \epsilon_5} a_1, \end{aligned} \tag{3.7}$$

where A and B are arbitrary constants. Then (3.7) takes the following form

$$a_5(s) = [\left(A - \int [\frac{k_1(s)k_3(s)}{k_2(s)} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$+ \left(B + \int [\frac{k_1(s)k_3(s)}{k_2(s)} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\sqrt{\epsilon_4\epsilon_5}a_1. \tag{3.8}$$

From the last equation of (3.4), the function a_4 is given by

$$a_4(s) = [\left(A - \int [\frac{k_1(s)k_3(s)}{k_2(s)} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$- \left(B + \int [\frac{k_1(s)k_3(s)}{k_2(s)} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}a_1. \tag{3.9}$$

From equation (3.9) with the first two equation in (3.4), leads to the following equation:

$$\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)' = [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$- \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}. \tag{3.10}$$

From equation (3.5), we have

$$\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]$$
$$= [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$+ \left(B + \int \frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds]\sqrt{\epsilon_4\epsilon_5}. \tag{3.11}$$

The equation (3.10) can be written by

$$\frac{\epsilon_5}{\sqrt{\epsilon_4\epsilon_5}\epsilon_3}\frac{k_1}{k_2}\left(\frac{k_1}{k_2}\right)' = [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$- \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right) \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\frac{k_1k_3}{k_2}.$$

If we integrate the above equation, we have

$$\frac{\epsilon_5}{\sqrt{\epsilon_4\epsilon_5}\epsilon_3}\frac{k_1^2}{k_2^2} = C - \left(\sqrt{\epsilon_4\epsilon_5}\right)^2 [\left(A - \int [\frac{k_1k_3}{k_2} \sin \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)^2$$
$$+ \left(B + \int [\frac{k_1k_3}{k_2} \cos \int \sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)^2] \tag{3.12}$$

where C is constant of integration. From equation (3.10), (3.11) and (3.12), we get

$$\frac{\sqrt{\epsilon_4\epsilon_5}\epsilon_5}{\epsilon_3}\left(\frac{k_1}{k_2}\right)^2 + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]^2 + \left[\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]^2 = C$$

$$(3.13)$$

Furthermore this constant C calculates as follows. From equation (3.13) together the three equations of (3.4), if the first Frenet vector V_1 is a timelike vector and V_i $(i = 2,3,4,5)$ is a spacelike vector, we have

$$C = \frac{a_3^2 + a_4^2 + a_5^2}{a_1^2} = \frac{1 - a_1^2}{a_1^2} = -\tanh^2\phi$$

Similarly, if the second Frenet vector V_2 is a timelike vector and V_i $(i = 1,3,4,5)$ is a spacelike vector, we have

$$C = \tan^2\phi$$

where we have used (2.1) and the fact that U is a unit vector field.

Conversely, assume that the condition (3.13) is satisfied for a curve γ. Let $\phi \in R$ be so that

$$C = \begin{cases} -\tanh^2\phi & \text{if } V_1 \text{ is a timelike curve} \\ \tan^2\phi & \text{if } V_1 \text{ is a spacelike curve} \end{cases}$$

Let define the unit vector U by if the first Frenet vector V_1 is a timelike vector

$$U = \cos h\phi \left[V_1 + \frac{k_1}{k_2} V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4 + \frac{1}{k_4}\left[\frac{k_1 k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] V_5 \right]$$

if V_1 is a timelike vector and

$$U = \cos\phi \left[V_1 + \frac{k_1}{k_2} V_3 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' V_4 + \frac{1}{k_4}\left[\frac{k_1 k_3}{k_2} + \left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] V_5 \right]$$

if V_1 is a spacelike vector. By taking account (3.13), a differentiation of U gives that $\frac{dU}{ds} = 0$, which it means that U is a constant vector field.On the other hand, the scalar product between the first Frent vector V_1 and U is

$$< V_1(s), U >= \begin{cases} \cos\phi = const, & \text{if } V_1 \text{ is a spacelike curve} \\ \cos h\phi = const, & \text{if } V_1 \text{ is a timelike curve} \end{cases}.$$

Thus γ is a nonnull inclined curve. $\square$

Theorem 3.2 *Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in Lorentzian 5-space L^5. Then γ is a nonnull inclined curve if and only if there exists a C^2-function f such that*

$$\begin{aligned} \epsilon_4 k_4 f(s) &= \epsilon_4 \frac{k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]', \\ \frac{1}{k_4}\frac{d}{ds} f(s) &= -\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'. \end{aligned} \qquad (3.14)$$

Proof Assume that γ is a nonnull inclined curve. A differentiation of (3.13) gives

$$\frac{\sqrt{\epsilon_4\epsilon_5}\epsilon_5}{\epsilon_3}\left(\frac{k_1}{k_2}\right)\left(\frac{k_1}{k_2}\right)' + \frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'$$

$$+\left\{\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right\}\left\{\left[\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]\right]'\right\}=0. \tag{3.15}$$

If we consider that $\epsilon_4 = \epsilon_5$, after some manipulations the equation (3.15) takes the following form:

$$\sqrt{\epsilon_4\epsilon_5}\frac{\epsilon_5 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'+\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]=0. \tag{3.16}$$

If we define $f = f(s)$ by

$$f(s)=\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2}+\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right].$$

Then the equation (3.16) writes as

$$f'(s)=-\sqrt{\epsilon_4\epsilon_5}\frac{\epsilon_5 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'.$$

Conversely, if (3.14) holds, we define a unit constant vector U by, if the first Frenet vector V_1 is a timelike vector

$$U=\cos h\phi\left[V_1+\frac{k_1}{k_2}V_3+\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'V_4+\frac{1}{k_4}\left[\frac{k_1 k_3}{k_2}+\left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]V_5\right]$$

and if the first Frenet vector V_1 is a spacelike vector

$$U=\cos\phi\left[V_1+\frac{k_1}{k_2}V_3+\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'V_4+\frac{1}{k_4}\left[\frac{k_1 k_3}{k_2}+\left[\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right]V_5\right].$$

We have that

$$< V_1(s), U >=\begin{cases}\cos\phi = const, & \text{if } V_1 \text{ is a spacelike curve}\\ \cos h\phi = const, & \text{if } V_1 \text{ is a timelike curve}\end{cases}.$$

that is γ is a nonnull inclined curve. □

Theorem 3.3 *Let $\gamma : I \subset R \to L^5$ be a unit speed nonnull curve in Lorentzian 5-space L^5. Then γ is a nonnull inclined curve if and only if the following condition is satisfied*

$$\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'=[\left(A-\int[\frac{k_1 k_3}{k_2}\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds$$
$$-\left(B+\int[\frac{k_1 k_3}{k_2}\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}. \tag{3.17}$$

for some constants A and B.

Proof Suppose that γ is a nonnull inclined curve. By using Theorem 3.2, let define $g(s)$ and $h(s)$ by

$$\psi(s)=\int^s k_4(u)du, \tag{3.18}$$

$$\begin{aligned} g(s) &= \frac{a_5}{a_1}\cos\sqrt{\epsilon_4\epsilon_5}\psi + \frac{\epsilon_5}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\sin\sqrt{\epsilon_4\epsilon_5}\psi + \int\left[\sqrt{\epsilon_4\epsilon_5}\frac{k_1k_3}{k_2}\sin\sqrt{\epsilon_4\epsilon_5}\psi\right]ds, \\ h(s) &= \frac{a_5}{a_1}\sin\sqrt{\epsilon_4\epsilon_5}\psi - \frac{\epsilon_5}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\cos\sqrt{\epsilon_4\epsilon_5}\psi + \int\left[\sqrt{\epsilon_4\epsilon_5}\frac{k_1k_3}{k_2}\cos\sqrt{\epsilon_4\epsilon_5}\psi\right]ds. \end{aligned} \tag{3.19}$$

If we differentiate equations (3.19) with respect to s and taking into account of (3.18), we obtain $\dfrac{dg}{ds} = 0$ and $\dfrac{dh}{ds} = 0$. Therefore, there exists constants A and B such that $g(s) = A$ and $h(s) = B$. By substituting into (3.19) and solving the resulting equations for $\dfrac{1}{\epsilon_3 k_3}\left(\dfrac{k_1}{k_2}\right)'$, we get

$$\begin{aligned} \frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)' &= [\left(A - \int[\frac{k_1k_3}{k_2}\sin\sqrt{\epsilon_4\epsilon_5}\psi]ds\right)\sin\sqrt{\epsilon_4\epsilon_5}\psi \\ &\quad - \left(B + \int[\frac{k_1k_3}{k_2}\cos\sqrt{\epsilon_4\epsilon_5}\psi]ds\right)\cos\sqrt{\epsilon_4\epsilon_5}\psi]\frac{\sqrt{\epsilon_4\epsilon_5}}{\epsilon_5}. \end{aligned} \tag{3.20}$$

Conversely, suppose that (3.17) holds. In order to apply Theorem 3.2 , we define

$$\begin{aligned} &\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] \\ &= [\left(A - \int[\frac{k_1k_3}{k_2}\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds \\ &\quad + \left(B + \int\frac{k_1k_3}{k_2}\cos\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds\right)\sin\int\sqrt{\epsilon_4\epsilon_5}k_4(s)ds]ds]\sqrt{\epsilon_4\epsilon_5} \end{aligned} \tag{3.21}$$

with $\psi(s) = \int^s k_{n-1}(u)du$, a direct differentiation of (3.17) gives

$$\left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]' = \left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] + \frac{\epsilon_4 k_1 k_3}{k_2}.$$

This shows the left condition in (3.17). Furthermore, a straightforward computation leads to

$$\frac{1}{\epsilon_4 k_4}\left[\frac{\epsilon_4 k_1 k_3}{k_2} + \left[\frac{1}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)'\right]'\right] = -\frac{\epsilon_4 k_4}{\epsilon_3 k_3}\left(\frac{k_1}{k_2}\right)',$$

which finish the proof. □

References

[1] Ali A.T., Inclined curves in Euclidean 5-space E^5, *Journal of Advanced Research in Pure Mathematics*, Vol.1,Issue.1, 2009, pp.15-22 Online ISSN:1943-2380.

[2] Ali A.T. and Lopez R., Some characterizations of cylindrical helices in E^n, *arXiv:0901.3325v1* [math.DG] 21 Jan. 2009.

[3] Ali A. . and Turgut M., Some characterizations of slant helices in the Euclidean space E^n, *arXiv:0904.1187v1*[math. DG].

[4] Altın A., Harmonic curvatures of nonnull curves and the helix in R^n_v, *Hacettepe Bulletin of Natural Sciences and Engineering*(Series B), 30(2001), 55-61.

[5] CamcıC., İlarslan K., Kula L. and Hacısalihoğlu H.H., Harmonic Curvatures and Generalized Helices in E^n, *Chaos, Solitons & Fractals* (2007), doi:10.1016/j.Chaos.2007.11.001.

[6] Ekmekci N., Hacısalihoğlu H. H., İlarslan K. Harmonic Curvatures in Lorentzian Space, *Bull. Malaysian Math. Sc. Soc.* (Second series)23(2000) 173-179.

Math.Combin.Book Ser. Vol.4(2012), 17-26

An Equation Related to θ-Centralizers in Semiprime Gamma Rings

M.F.Hoque and H.O.Roshid

Department of Mathematics, Pabna Science and Technology University, Pabna-6600, Bangladesh

A.C.Paul

Department of Mathematics, Rajshahi University, Rajshahi-6205,Bangladesh

E-mail: fazlul_math@yahoo.co.in, acpaulrubd_math@yahoo.com

Abstract: Let M be a 2-torsion free semiprime Γ-ring satisfying a certain assumption and θ be an endomorphism on M. Let $T : M \to M$ be an additive mapping such that

$$2T(a\alpha b\beta a) \quad = T(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta T(a) \tag{1}$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then we prove that T is a θ-centralizer.

Key Words: Semiprime Γ-ring, left centralizer, centralizer, Jordan centralizer, left θ-centralizer, θ-centralizer, Jordan θ-centralizer.

AMS(2010): 16N60,16W25,16U80

§1. Introduction

Let M and Γ be additive Abelian groups. If there exists a mapping $(x, \alpha, y) \to x\alpha y$ of $M \times \Gamma \times M \to M$, which satisfies the conditions

(i) $x\alpha y \in M$;
(ii) $(x + y)\alpha z$=$x\alpha z$+$y\alpha z$, $x(\alpha + \beta)z$=$x\alpha z$+$x\beta z$, $x\alpha(y + z)$=$x\alpha y$+$x\alpha z$;
(iii) $(x\alpha y)\beta z$=$x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$,

then M is called a Γ-ring.

Every ring M is a Γ-ring with M=Γ. However a Γ-ring need not be a ring. Gamma rings, more general than rings, were introduced by Nobusawa[13]. Bernes[1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa.

Let M be a Γ-ring. Then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U(U\Gamma M \subset U)$. If U is both a left and a right ideal , then we say U is an ideal of M. Suppose again that M is a Γ-ring. Then M is said to be a 2-torsion free if $2x$=0 implies x=0 for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for any ideals A and B of M,

[1]Received October 16, 2012. Accepted November 28, 2012.

$A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b$=(0) with $a, b \in M$, implies a=0 or b=0 and semiprime if $a\Gamma M\Gamma a$=(0) with $a \in M$ implies a=0. Furthermore, M is said to be commutative Γ-ring if $x\alpha y$=$y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover,the set $Z(M)$ =$\{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma, y \in M\}$ is called the centre of the Γ-ring M.

If M is a Γ-ring, then $[x, y]_\alpha$=$x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x, y \in M$ and $\alpha \in \Gamma$. We make the basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$$

for all $x, y.z \in M$ and $\alpha, \beta \in \Gamma$. We consider the following assumption:

(A) $x\alpha y\beta z$=$x\beta y\alpha z$, for all $x, y, z \in M$, and $\alpha, \beta \in \Gamma$.

According to the assumption (A), the above two identities reduce to

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$$
$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta,$$

which we extensively used. An additive mapping $T : M \to M$ is a left(right) centralizer if

$$T(x\alpha y) = T(x)\alpha y(T(x\alpha y) = x\alpha T(y))$$

holds for all $x, y \in M$ and $\alpha \in \Gamma$. A centralizer is an additive mapping which is both a left and a right centralizer. For any fixed $a \in M$ and $\alpha \in \Gamma$, the mapping $T(x) = a\alpha x$ is a left centralizer and $T(x) = x\alpha a$ is a right centralizer. We shall restrict our attention on left centralizer, since all results of right centralizers are the same as left centralizers. An additive mapping $D : M \to M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$, and $\alpha \in \Gamma$ and is called a Jordan derivation if $D(x\alpha x) = D(x)\alpha x + x\alpha D(x)$ for all $x \in M$ and $\alpha \in \Gamma$. An additive mapping $T : M \to M$ is Jordan left(right) centralizer if

$$T(x\alpha x) = T(x)\alpha x(T(x\alpha x) = x\alpha T(x))$$

for all $x \in M$, and $\alpha \in \Gamma$.

Every left centralizer is a Jordan left centralizer but the converse is not ingeneral true.

An additive mappings $T : M \to M$ is called a Jordan centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha y + y\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. Every centralizer is a Jordan centralizer but Jordan centralizer is not in general a centralizer.

Bernes[1], Luh [8] and Kyuno[7] studied the structure of Γ-rings and obtained various generalizations of corresponding parts in ring theory. Borut Zalar [15] worked on centralizers of semiprime rings and proved that Jordan centralizers and centralizers of this rings coincide. Joso Vukman[12, 13, 14] developed some remarkable results using centralizers on prime and semiprime rings. Vukman and Irena [11] proved that if R is a 2-tortion free semiprime ring and $T : R \to R$ is an additive mapping such that $2T(xyx) = T(x)yx + xyx$ holds for all $x, y \in R$,

then T is a centralizer. Y.Ceven [2] worked on Jordan left derivations on completely prime Γ-rings. He investigated the existence of a nonzero Jordan left derivation on a completely prime Γ-ring that makes the Γ-ring commutative with an assumption. With the same assumption, he showed that every Jordan left derivation on a completely prime Γ-ring is a left derivation on it.

In [3], M.F. Hoque and A.C Paul have proved that every Jordan centralizer of a 2-torsion free semiprime Γ-ring is a centralizer. There they also gave an example of a Jordan centralizer which is not a centralizer.

In [4], M.F. Hoque and A.C Paul have proved that if M is a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and if $T : M \to M$ is an additive mapping such that

$$T(x\alpha y\beta x) \quad = x\alpha T(y)\beta x$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, then T is a centralizer. Also, they have proved that T is a centralizer if M contains a multiplicative identity 1.

In [5], M.F. Hoque and A.C Paul have proved that if M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and let $T : M \to M$ be an additive mapping such that

$$2T(a\alpha b\beta a) \quad = T(a)\alpha b\beta a + a\alpha b\beta T(a)$$

holds for all pairs $a, b \in M$, and $\alpha, \beta \in \Gamma$. Then T is a centralizer.

In [10], Z.Ullah and M.A.Chaudhary have proved that every Jordan θ-centralizer of a 2-torsion free semiprime Γ-ring is a θ-centralizer.

In [6] M.F. Hoque and A.C Paul have given an example of a Jordan θ-centralizer which is not a θ-centralizer and another two examples which was ensure that θ-centralizer and a Jordan θ-centralizer exist in Γ-ring. There they also have proved that if M be a 2-torsion free semiprime Γ-ring satisfying a certain assumption and θ be an endomorphism of M. Let $T : M \to M$ be an additive mapping such that

$$T(x\alpha y\beta x) \quad = \theta(x)\alpha T(y)\beta\theta(x)$$

holds for all $x, y \in M$, and $\alpha, \beta \in \Gamma$. Then T is a θ-centralizer.

In this paper we study certain results using the concept of θ-centralizer on semiprime gamma ring.

§2. The θ-Centralizers of Semiprime Gamma Rings

In this section we have given the following definitions:

Let M be a 2-torsion free semiprime Γ-ring and let θ be an endomorphism of M. An additive mapping $T : M \to M$ is a left(right) θ-centralizer if $T(x\alpha y) = T(x)\alpha\theta(y)(T(x\alpha y) = \theta(x)\alpha T(y))$ holds for all $x, y \in M$ and $\alpha \in \Gamma$. If T is a left and a right θ-centralizer, then it is natural to call T a θ-centralizer.

Let M be a Γ-ring and let $a \in M$ and $\alpha \in \Gamma$ be fixed element. Let $\theta : M \to M$ be an endomorphism. Define a mapping $T : M \to M$ by $T(x)a\alpha\theta(x)$. Then it is clear that T is a left θ-centralizer. If $T(x) = \theta(x)\alpha a$ is defined, then T is a right θ-centralizer.

An additive mapping $T : M \to M$ is Jordan left(right) θ-centralizer if

$$T(x\alpha x) = T(x)\alpha\theta(x)(T(x\alpha x) = \theta(x)\alpha T(x))$$

holds for all $x \in M$ and $\alpha \in \Gamma$. It is obivous that every left θ-centralizer is a Jordan left θ-centralizer but in general Jordan left θ-centralizer is not a left θ-centralizer.

Let M be a Γ-ring and let θ be an endomorphism on M. An additive mapping $T : M \to M$ is called a Jordan θ-centralizer if $T(x\alpha y + y\alpha x) = T(x)\alpha\theta(y) + \theta(y)\alpha T(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$. It is clear that every θ-centralizer is a Jordan θ-centralizer but the converse is not in general a θ- centralizer.

An additive mapping $D : M \to M$ is called a (θ, θ)-derivation if $D(x\alpha y) = D(x)\alpha\theta(y) + \theta(x)\alpha D(y)$ holds for all $x, y \in M$ and $\alpha \in \Gamma$ and is called a Jordan (θ, θ)-derivation if $D(x, x) = D(x)\alpha\theta(x) + \theta(x)\alpha D(x)$ holds for all $x \in M$ and $\alpha \in \Gamma$.

For proving our main results, we need the following Lemmas:

Lemma 2.1([4]) *Suppose M is a semiprime Γ-ring satisfying the assumption (A). Suppose that the relation $x\alpha a\beta y + y\alpha a\beta z = 0$ holds for all $a \in M$, some $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Then $(x + z)\alpha a\beta y = 0$ is satisfied for all $a \in M$ and $\alpha, \beta \in \Gamma$.*

Lemma 2.2 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and θ be an endomorphism of M. Suppose that $T : M \to M$ is an additive mapping such that*

$$2T(a\alpha b\beta a) \quad = T(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta T(a)$$

holds for all pairs $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then $2T(a\gamma a) = T(a)\gamma\theta(a) + \theta(a)\gamma T(a)$.

Proof Putting $a + c$ for a in (1)(linearization), we have

$$\begin{aligned} 2T(a\alpha b\beta c + c\alpha b\beta a) \quad &= \quad T(a)\alpha\theta(b)\beta\theta(c) + T(c)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(c)\alpha\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\beta T(c) \end{aligned} \tag{2}$$

Putting $c = a\gamma a$ in (2), we have

$$\begin{aligned} 2T(a\alpha b\beta a\gamma a + a\gamma a\alpha b\beta a) \quad &= \quad T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) + T(a\gamma a)\alpha\theta(b)\beta\theta(a) \\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\beta T(a\gamma a) \end{aligned} \tag{3}$$

Replacing b by $a\gamma b + b\gamma a$ in (1), we have

$$\begin{aligned} 2T(a\alpha a\gamma b\beta a + a\alpha b\gamma a\beta a) \quad &= \quad T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a) + T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a) \\ &\quad +\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a) + \theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a) \end{aligned} \tag{4}$$

Subtracting (4) from (3), using assumption (A), gives

$$(T(a\gamma a) - T(a)\gamma\theta(a))\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha(b)\beta(T(a\gamma a) - \theta(a)\gamma T(a)) \quad = 0.$$

Taking $\theta(x) = T(a\gamma a) - T(a)\gamma\theta(a)$, $y = a$, $c = b$ and $\theta(z) = T(a\gamma a) - \theta(a)\gamma T(a)$. Then the above relation becomes $\theta(x)\alpha\theta(c)\beta\theta(y) + \theta(y)\alpha\theta(c)\beta\theta(z) = 0$. Thus using Lemma 2.1, we get $(\theta(x) + \theta(z))\alpha\theta(c)\beta\theta(y) = 0$. Hence

$$(2T(a\gamma a) - T(a)\gamma\theta(a) - \theta(a)\gamma T(a))\alpha\theta(b)\beta\theta(a) \quad = 0.$$

If we take

$$A(a) \quad = 2T(a\gamma a) - T(a)\gamma\theta(a) - \theta(a)\gamma T(a),$$

then the above relation becomes

$$A(a)\alpha\theta(b)\beta\theta(a) \quad = 0$$

Using the assumption (A), we obtain

$$A(a)\beta\theta(b)\alpha\theta(a) \quad = 0 \tag{5}$$

Replacing b by $a\alpha b\gamma A(a)$ in (5), we have

$$A(a)\beta\theta(a)\alpha\theta(b)\gamma A(a)\alpha\theta(a) \quad = 0$$

Again using the assumption (A), we have

$$A(a)\alpha\theta(a)\beta\theta(b)\gamma A(a)\alpha\theta(a) \quad = 0$$

By the semiprimeness of M, we have

$$A(a)\alpha\theta(a) = 0 \tag{6}$$

Similarly, if we multiplying (5) from the left by $\theta(a)\alpha$ and from the right side by $\gamma A(a)$, we obtain

$$\theta(a)\alpha A(a)\beta\theta(b)\alpha\theta(a)\gamma A(a) \quad = 0$$

Using the assumption (A),

$$\theta(a)\alpha A(a)\beta\theta(b)\gamma\theta(a)\alpha A(a) \quad = 0$$

and by the semiprimeness, we obtain

$$\theta(a)\alpha A(a) \quad = 0 \tag{7}$$

Replacing a by $a+b$ in (6)(linearization), we have

$$A(a)\alpha\theta(b) + A(b)\alpha\theta(a) + B(\theta(a), \theta(b))\alpha\theta(a) + B(\theta(a), \theta(b))\alpha\theta(b) = 0,$$

where

$$B(\theta(a), \theta(b)) \quad = 2T(a\gamma b + b\gamma a) - T(a)\gamma\theta(b) - T(b)\gamma\theta(a) - \theta(a)\gamma T(b) - \theta(b)\gamma T(a)$$

Replacing a by $-a$ in the above relation and comparing these relation, and by using the 2-torsion freeness of M, we arrive at

$$A(a)\alpha\theta(b) + B(\theta(a), \theta(b))\alpha\theta(a) \quad = 0 \tag{8}$$

Right multiplication of the above relation by $\beta A(a)$ along with (7) gives

$$A(a)\alpha\theta(b)\beta A(a)+B(\theta(a),\theta(b))\alpha\theta(a)\beta A(a) \quad =0$$

Since $\theta(a)\beta A(a)=0$, for all $\beta\in\Gamma$, we have

$$B(\theta(a),\theta(b))\alpha\theta(a)\beta A(a) \quad =0$$

This implies that

$$A(a)\alpha\theta(b)\beta A(a) \quad =0$$

By semiprimeness, we have

$$A(a) \quad =0$$

Thus we have

$$2T(a\gamma a)=T(a)\gamma\theta(a)+\theta(a)\gamma T(a).$$ □

Lemma 2.3 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and θ be an endomorphism of M. Let $T:M\to M$ be an additive mapping such that*

$$2T(a\alpha b\beta a) \quad =T(a)\alpha\theta(b)\beta\theta(a)+\theta(a)\alpha\theta(b)\beta T(a)$$

holds for all pairs $a,b\in M$ and $\alpha,\beta\in\Gamma$. Then

$$[T(a),\theta(a)]_\alpha \quad =0 \tag{9}$$

Proof Replacing a by $a+b$ in relation (9)(linearization) gives

$$2T(a\gamma b+b\gamma a) \quad =T(a)\gamma\theta(b)+T(b)\gamma\theta(a)+\theta(a)\gamma T(b)+\theta(b)\gamma T(a) \tag{10}$$

Replacing b with $2a\alpha b\beta a$ in (11) and use (1), we obtain

$$\begin{aligned}4T(a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a) &= 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+2T(a\alpha b\beta a)\gamma\theta(a)\\ &\quad +2\theta(a)\gamma T(a\alpha b\beta a)+2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a)\\ &= 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a)\\ &\quad +\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)+\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)+2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a)\end{aligned}$$

$$\begin{aligned}4T(a\gamma a\alpha b\beta a+a\alpha b\beta a\gamma a) &= 2T(a)\gamma\theta(a)\alpha\theta(b)\beta\theta(a)+T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a)\\ &\quad +\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)+\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\\ &\quad +\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\\ &\quad +2\theta(a)\alpha\theta(b)\beta\theta(a)\gamma T(a)\end{aligned} \tag{11}$$

Comparing (4) and (12), we arrive at

$$\begin{aligned} T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) &+ \theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a) \\ &- \theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) = 0 \end{aligned} \tag{12}$$

Putting $b\gamma a$ for b in the above relation, we have

$$\begin{aligned} &T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a) \\ &\quad -\theta(a)\alpha\theta(b)\gamma\theta(a)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\gamma\theta(a)\beta\theta(a) = 0 \end{aligned} \tag{13}$$

Right multiplication of (13) by $\gamma\theta(a)$ gives

$$\begin{aligned} &T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) \\ &\quad -\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) = 0 \end{aligned} \tag{14}$$

Subtracting (14) from (15) and using assumption (A), we get

$$\theta(a)\gamma\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - \theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) = 0 \tag{15}$$

The substitution $T(a)\alpha b$ for b in (16), we have

$$\theta(a)\gamma\theta(a)\gamma T(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha - \theta(a)\gamma T(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) = 0 \tag{16}$$

Left multiplication of (16) by $T(a)\alpha$ gives

$$T(a)\alpha\theta(a)\gamma\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - T(a)\alpha\theta(a)\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) = 0 \tag{17}$$

Subtracting (17) from (18), we arrive at

$$[T(a),\theta(a)\gamma\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) = 0$$

In the above relation let

$$\theta(x) = [T(a),\theta(a)\gamma\theta(a)]_\alpha,\ \theta(y) = [T(a),\theta(a)]_\alpha,\ \theta(z) = -[T(a),\theta(a)]_\alpha\gamma\theta(a)$$

and $c = b$. Then we have

$$\theta(x)\gamma\theta(c)\beta\theta(y) + \theta(y)\gamma\theta(c)\beta\theta(z) = 0$$

Thus from Lemma 2.1, we have

$$\begin{aligned} &(\theta(x)+\theta(z))\gamma\theta(c)\beta\theta(y) = 0 \\ &\Rightarrow ([T(a),\theta(a)\gamma\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(a))\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \end{aligned}$$

This implies that

$$\begin{aligned} &([T(a),\theta(a)]_\alpha\gamma\theta(a) + \theta(a)\gamma[T(a),\theta(a)]_\alpha - [T(a),\theta(a)]_\alpha\gamma\theta(a))\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \\ &\Rightarrow \theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha = 0 \end{aligned}$$

Putting $b = b\alpha a$ in the above relation, we have

$$\theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\alpha\theta(a)\beta[T(a),\theta(a)]_\alpha = 0$$
$$\Rightarrow \theta(a)\gamma[T(a),\theta(a)]_\alpha\alpha\theta(b)\beta\theta(a)\gamma[T(a),\theta(a)]_\alpha = 0$$

using the assumption (A). By the semiprimeness of M, we obtain

$$\theta(a)\gamma[T(a),\theta(a)]_\alpha \quad = 0 \tag{18}$$

Putting $a\gamma b$ for b in the relation (13), we obtain

$$\begin{aligned}&T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a)\\&-\theta(a)\alpha\theta(a)\gamma\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma T(a)\alpha\theta(a)\gamma\theta(b)\beta\theta(a) = 0\end{aligned} \tag{19}$$

Left multiplication of (13) by $\theta(a)\gamma$, we have

$$\begin{aligned}&\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a)\gamma\theta(a) + \theta(a)\gamma\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\\&-\theta(a)\gamma\theta(a)\alpha\theta(b)\beta T(a)\gamma\theta(a) - \theta(a)\gamma\theta(a)\gamma T(a)\alpha\theta(b)\beta\theta(a) = 0\end{aligned} \tag{20}$$

Subtracting (21) from (20), and using assumption (A), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a) - \theta(a)\gamma[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a) \quad = 0$$

Using (19) in the above relation, we obtain

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a) \quad = 0 \tag{21}$$

Putting $b\alpha T(a)$ for b in (22), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\alpha T(a)\beta\theta(a)\gamma\theta(a) \quad = 0 \tag{22}$$

Right multiplication of (22) by $\alpha T(a)$ gives

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta\theta(a)\gamma\theta(a)\alpha T(a) \quad = 0 \tag{23}$$

Subtracting (24) from (23) and using assumption (A), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)\gamma\theta(a)]_\alpha \quad = 0$$

The above relation can be rewritten and using (19), we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0$$

Putting $a\alpha b$ for b in the above relation, we obtain

$$[T(a),\theta(a)]_\alpha\gamma\theta(a)\alpha\theta(b)\beta[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0$$

By semiprimeness of M, we have

$$[T(a),\theta(a)]_\alpha\gamma\theta(a) \quad = 0 \tag{24}$$

Replacing a by $a+b$ in (19) and then using (19) gives

$$\begin{aligned}&\theta(a)\gamma[T(a),\theta(b)]_\alpha+\theta(a)\gamma[T(b),\theta(a)]_\alpha+\theta(a)\gamma[T(b),\theta(b)]_\alpha\\&+\theta(b)\gamma[T(a),\theta(a)]_\alpha+\theta(b)\gamma[T(a),\theta(b)]_\alpha+\theta(b)\gamma[T(b),\theta(a)]_\alpha=0\end{aligned}$$

Replacing a by $-a$ in the above relation and comparing the relation so obtained with the above relation, we have

$$\theta(a)\gamma[T(a),\theta(b)]_\alpha+\theta(a)\gamma[T(b),\theta(a)]_\alpha+\theta(b)\gamma[T(a),\theta(a)]_\alpha \quad =0 \tag{25}$$

Left multiplication of (26) by $[T(a),\theta(a)]_\alpha\beta$ and then use (25), we have

$$[T(a),\theta(a)]_\alpha\beta\theta(b)\gamma[T(a),\theta(a)]_\alpha \quad =0$$

By semiprimeness of M, we have

$$[T(a),\theta(a)]_\alpha \quad =0$$

Hence the relation (10) follows. □

Theorem 2.1 *Let M be a 2-torsion free semiprime Γ-ring satisfying the assumption (A) and and θ be an endomorphism of M. Let $T : M \to M$ be an additive mapping such that $2T(a\alpha b\beta a) = T(a)\alpha\theta(b)\beta\theta(a) + \theta(a)\alpha\theta(b)\beta T(a)$ holds for all pairs $a, b \in M$ and $\alpha, \beta \in \Gamma$. Then T is a θ-centralizer.*

Proof The relation (9) in Lemma 2.2 and the relation (10) in Lemma 2.3 give

$$T(a\alpha a) = T(a)\alpha\theta(a) \quad \text{and} \quad T(a\alpha a) = \theta(a)\alpha T(a)$$

since M is a 2-torsion free. Hence T is a left and also a right Jordan θ-centralizers. By Theorem 2.1 in [3], it follows that T is a left and also a right θ-centralizer which completes the proof of the theorem. □

References

[1] W.E.Barnes, On the Γ-rings of Nobusawa, *Pacific J.Math.*, 18(1966), 411-422.

[2] Y.Ceven, Jordan left derivations on completely prime gamma rings, *C.U.Fen-Edebiyat Fakultesi,Fen Bilimleri Dergisi* (2002) Cilt 23 Sayi 2.

[3] M.F.Hoque and A.C.Paul, On centralizers of semiprime gamma rings, *International Mathematical Forum*, Vol.6(2011), No.13,627-638.

[4] M.F.Hoque and A.C.Paul, Centralizers on semiprime gamma rings, *Italian J. Of Pure and Applied Mathematics*(To appear).

[5] M.F.Hoque and A.C.Paul, An equation related to centralizers in semiprime gamma rings, *Annals of Pure and Applied Mathematics*, Vol.1, No.1,84-90,2012.

[6] M.F.Hoque and A.C.Paul, The θ-centralizers of semiprime gamma rings(communicated).

[7] S.Kyuno, On prime gamma ring, *Pacific J.Math.*, 75(1978), 185-190.
[8] L.Luh, On the theory of simple gamma rings, *Michigan Math.J.*, 16(1969), 65-75.
[9] N.Nobusawa, On the generalization of the ring theory, *Osaka J. Math.*, 1(1964), 81-89.
[10] Z.Ullah and M.A.Chaudhary, On K-centralizers of semiprime gamma rings, *Int. J. of Algebra*, Vol.6, 2012, No.21, 1001-1010.
[11] J.Vukman and I.Kosi-Ulbl, On centralizers of semiprime rings, Aequationes Math. 66, 3(2003), 277-283.
[12] J.Vukman, Centralizers in prime and semiprime rings, *Comment. Math. Univ. Carolinae*, 38(1997), 231-240.
[13] J.Vukman, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae*, 40,3(1999), 447-456.
[14] J.Vukman, Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, 42,2(2001), 237-245.
[15] B.Zalar, On centralizers of semiprime rings, *Comment.Math. Univ. Carolinae*, 32(1991),609-614.

Math.Combin.Book Ser. Vol.4(2012), 27-32

Homomorphism of Fuzzy Bigroup

Akinola L.S.

(Department of Mathematical and Computer Sciences, Fountain University, Osogbo, Nigeria)

Agboola A.A.A. and Adeniran J. O.

(Department of Mathematics, University of Agriculture, Abeokuta, Nigeria)

E-mail: lukmanakinola@gmail.com, aaaola2003@yahoo.com, adeniranoj@unaab.edu.ng

Abstract: In this paper, we study some aspects of homomorphism of fuzzy bigroup using the concept of restricted fuzzy bigroup we introduced in [1]. We define weakly fuzzy bigroup homomorphism and study its properties. We also give the fuzzy bigroup equivalent concepts of I, II, III, IV- fuzzy group homomorphisms and study the relationship between I and II fuzzy bigroup homomorphisms.

Key Words: Bigroups, fuzzy bigroups, restricted fuzzy bigroup, weakly fuzzy bigroup homomorphism, fuzzy bigroups homomorphism.

AMS(2010): 03E72, 20D25

§1. Introduction

There are several different fuzzy approaches to homomorphisms existing in literature concerning fuzzy algebra generally and fuzzy groups in particular. Bělohlàvek and Vychodil [2] studied ordinary homomorphisms of algebras which are compatible with fuzzy equalities. Jelana et al [4] studied fuzzy homomorphisms of algebras. The study of fuzzy homomorphism between two groups was initiated by Chakrabonty and Khare [3]. The concept was further studied by Suc-Yun Li et. al. [9]. Many other researchers have also studied different aspects of fuzzy group homomorphisms. See for instance [13] The notion of bigroup was first introduced by P.L.Maggu [5]. This idea was futher studied by Vasantha and Meiyappan [11]. These authors gave modifications of some results earlier proved by Maggu. Meiyappan [6,10] introduced fuzzy bigroup of a bigroup and studied some of its properties. Akinola and Agboola [1] also studied further properties of fuzzy bigroup.

In this paper, using the concept of restricted fuzzy bigroup we introduced in [1], We define weakly fuzzy bigroup homomorphism and study its properties. We also define the concept of I, II, III, IV- fuzzy bigroup homomorphisms and study the relationships between I- fuzzy bigroup homomorphism and II- fuzzy bigroup homomorphism.

[1]Received July 12, 2012. Accepted November 30, 2012.

§2. Preliminary Results

Definition 2.1([13]) *Let G and $G^{'}$ be two groups, and let λ and μ be separately fuzzy subgroups of G and $G^{'}$. If there exists a mapping $\phi : G \to G^{'}$; $f : \lambda(G) \to \mu(G')$ satisfying:*

(i) $G \overset{\phi}{\sim} G'$;

(ii) $f(\lambda(xy)) = \mu(\phi(x)\phi(y))$ *for any* $x,\ y \in G$;

then λ and μ are said to be weakly homomorphic, and it is denoted as $\lambda(\overset{\phi,f}{\sim})\mu$.

Definition 2.2([12]) *Let X be non empty set. A fuzzy subset μ in the set X has the sup property if for any subset A of the set X there exists $x_0 \in A$ such that*

$$\mu(x_0) = sup\{\mu(x) : x \in A\}.$$

Definition 2.2 is applicable for a group G and a fuzzy subgroup μ of G.

Definition 2.3([9]) *Let η be a homomorphism (isomorphism) from G onto $G^{'}$. λ and μ are separately fuzzy subgroups of G and $G^{'}$. If $\mu = \eta(\lambda)$, then we say λ and μ are $I-$homomorphic (isomorphic).*

Definition 2.4([9]) *Let λ and μ be fuzzy subgroups of G and $G^{'}$. If for any $\alpha \in [0,1]$, $\lambda_\alpha \sim \mu_\alpha$ $(\lambda_\alpha \cong \mu_\alpha)$, then we say λ and μ are II-homomorphic(isomorphic).*

Definition 2.5([9]) *Let θ be a homomorphism from G onto $G^{'}$. λ and μ are separate fuzzy subgroups of G and $G^{'}$. If $f : \lambda(G) \to \mu(G^{'})$ is such that $f(\lambda(x)) = \mu(\theta(x))$, then we say λ and μ are III-homomorphic. If θ is an isomorphism and f is a monomorphism we call λ and μ III-isomorphic.*

Definition 2.6([9]) *Let λ and μ be fuzzy subgroups of G and $G^{'}$. Then λ and μ are said to be $IV-$homomorphic(isomorphic) if:*

(i) *for any λ_α, $\alpha \in [0,1]$, there exists at least one μ_β such that $\lambda_\alpha \sim \mu_\beta$ $(\lambda_\alpha \cong \mu_\beta)$ and for any $\lambda_\gamma \supseteq \lambda_\alpha$ there exists $\mu_\delta \supseteq \mu_\beta$ $(\lambda_\alpha \cong \mu_\beta)$ such that $\lambda_\gamma \sim \mu_\beta$;*

(ii) *for any μ_β, $\beta \in [0,1]$ there exists at least one λ_α such that $\lambda_\gamma \sim \mu_\beta$ $(\lambda_\gamma \cong \mu_\beta)$, and for any $\mu_\delta \supseteq \mu_\beta$, there is a $\lambda_\gamma \supseteq \lambda_\alpha$, such that $\lambda_\gamma \sim \mu_\beta(\lambda_\gamma \cong \mu_\beta)$.*

Proposition 2.7([13]) *Let $G(\overset{\theta}{\sim})\ G'$, $f : \lambda(G) \to \mu(G')$. Then $\lambda(\overset{\theta,f}{\sim})\mu$ if and only if $f(\lambda(x)) = \mu(\theta(x))$.*

Proposition 2.8([9]) *Suppose $\lambda(\theta, f)\mu$: (i) If $\lambda(x) < \lambda(y)$, then $\mu(\theta(x)) \le \mu(\theta(y))$, $x,\ y \in G$; (ii) If $\mu(x^{'}) < \mu(y^{'})$, then $\lambda(x) \le \lambda(y)$, $x^{'}, y^{'} \in\ \theta(G); x, y \in G;\ \theta(x) = x^{'},\ \theta(y) = y^{'}$.*

Proposition 2.9([9]) (i) *If λ and μ are $I-$homomorphic and λ has the sup-property, then they are II-homomorphic.*

(ii) *If λ and μ are $I-$isomorphic, then they are $II-$isomorphic.*

Proposition 2.10([9]) *Let λ and μ be $ll-$homomorphic. λ has the sup-property. Then λ and μ are $l-$homomorphic iff they have a homomorphism η from G onto G' such that $\eta(\lambda_\alpha) = \mu_\alpha$.*

Definition 2.11([5]) *A set $(G, +, \cdot)$ with two binary operations $''+''$ and $''\cdot''$ is called a bi-group if there exist two proper subsets G_1 and G_2 of G such that*

(*i*) $G = G_1 \cup G_2$;
(*ii*) $(G_1, +)$ *is a group;*
(*iii*) $(G_2, \cdot)$ *is a group.*

Definition 2.12([6]) *Let $(G, +, \cdot)$ and $(H, \oplus, \circ)$ be any two bigroups where $G = G_1 \cup G_2$, and $H = H_1 \cup H_2$, G_1, G_2, are fuzzy subgroups of G, $H = H_1$, H_2 are fuzzy subgroups of H. The map $f : G \to H$ is said to be a bigroup homomorphism if f restricted to G_1 $(i.e. f \backslash G_1)$ is a group homomorphism from G_1 to H_1 and f restricted to $G_2 (i.e. f \backslash G_2)$ is a group homomorphism from G_2 to H_2.*

Definition 2.13([1]) *Let $(G, +, \cdot)$ be a bi-group . Let $(G_1, +)$, $(G_2, \cdot)$ be the constituting groups of G. Define $\gamma_{G_1} : G \to [0, 1]$ as*

$$\gamma_{G_1}(x) = \begin{cases} \alpha > 0 & if\ x \in G_1 \cap G, \\ \\ 0 & if\ x \notin G_1 \cap G \end{cases}$$

We call γ_{G_1} a G_1 restricted fuzzy subgroup of G if it satisfies the conditions of Rosenfeld [8] fuzzy subgroup. Similarly we define $\gamma_{G_2} : G \to [0, 1]$ as

$$\gamma_{G_2}(x) = \begin{cases} \beta > 0 & if\ x \in G_2 \cap G, \\ \\ 0 & if\ x \notin G_2 \cap G \end{cases}$$

which we also call γ_{G_2} a G_2 restricted fuzzy subgroup of G under the same situation. Then, $\gamma : G \to [0, 1]$ where $\gamma = \gamma_{G_1} \cup \gamma_{G_2}$ is a fuzzy bigroup of G.

§3. Main Results

Definition 3.1 *Let $(G, +, \cdot)$ and $(H, \oplus, \circ)$ be bigroups. Let $\gamma_G = \gamma_{G_1} \cup \gamma_{G_2}$ and $\rho_H = \rho_{H_1} \cup \rho_{H_2}$ be separate fuzzy sub bigroup of G and H respectively. If $\theta : G \to H$ is a bigroup homomorphism then the mapping $\phi : \gamma_G \to \rho_H$ is said to be weakly fuzzy homomorphic if for any $x, y, \in G$, $\phi(\gamma_G(xy)) = \rho_H(\theta(x)\theta(y))$. It is denoted as $\gamma_G(\overset{\theta,\ \phi}{\sim})\rho_H$.*

Theorem 3.2 *Let $\gamma_G(\overset{\theta,\ \phi}{\sim})\rho_H$ where $\theta : G \to H$ is a bigroup homomorphism, then $\phi \mid G_1 : \gamma_{G_1} \to \rho_{H_1}$ and $\phi \mid G_2 : \gamma_{G_2} \to \rho_{H_2}$ are both restricted weakly fuzzy group homomorphisms.*

Proof Suppose that $\gamma_G(\overset{\theta,\ \phi}{\sim})\rho_H$ where $\theta : G \to H$ and $\phi : \gamma_G \to \rho_H$, then

$$\forall x,\ y, \in G,\ \phi(\gamma_G(xy)) = \rho_H(\theta(x)\theta(y))$$

which implies that

$$\phi(\gamma_{G_1} \cup \gamma_{G_2})(xy)) = (\rho_{H_1} \cup \rho_{H_2})(\theta(x)\theta(y)).$$

If $x, y \in G_1 \cap G_2^c$, then

$$\phi(\gamma_{G_1} \cup \gamma_{G_2})(xy)) = \phi(\gamma_{G_1}(xy)) = \rho_{H_1}((\theta(x)\theta(y))$$

which shows that $\phi \mid G_1$ is a weakly restricted fuzzy homomorphism from $\gamma_{G_1} \to \rho_{H_1}$. Similarly, if $x, y \in G_1^c \cap G_2$, then

$$\phi(\gamma_{G_1} \cup \gamma_{G_2})(xy)) = \phi(\gamma_{G_2}(xy)) = \rho_{H_2}((\theta(x)\theta(y))$$

which also shows that $\phi \mid G_2$ is a weakly restricted fuzzy homomorphism from $\gamma_{G_2} \to \rho_{H_2}$. □

Theorem 3.3 *Suppose that $\phi \mid G_1 : \gamma_{G_1} \to \rho_{H_1}$ and $\phi \mid G_2 : \gamma_{G_2} \to \rho_{H_2}$ are both restricted weakly fuzzy group homomorphisms and let $\theta : G \to H$ be a bigroup homomorphism then $\phi : \gamma_G \to \rho_H$ is a weakly fuzzy bigroup homomorphism if G_1 and G_2 are distinct subgroups of the bigroup G.*

Proof The theorem is a converse of theorem 3.2. Suppose that $\phi \mid G_1 : \gamma_{G_1} \to \rho_{H_1}$ is a restricted weakly fuzzy group homomorphism then, for $\forall\ x, y\ \in G_1$

$$\phi(\gamma_G(xy)) = \phi(\gamma_{G_1} \cup \gamma_{G_2})(xy)) = \phi(\gamma_{G_1}(xy)) = \rho_{H_1}(\theta(xy)) = \rho_{H_1}(\theta(x)\theta(y)) = \rho_{H_1}(\theta(x)\theta(y)).$$

Since G_1 and G_2 are distinct fuzzy subgroup,

$$x, y \notin G_2 \Rightarrow \phi(\gamma_{G_2}(xy)) = \phi(0) = \rho_{H_2}\theta(0) = \rho_{H_2}((\theta(0)\theta(0)).$$

Similarly, for $\forall\ x, y\ \in G_2$,

$$\phi(\gamma_G(xy)) = \phi(\gamma_{G_1} \cup \gamma_{G_2})(xy)) = \phi(\gamma_{G_2}(xy)) = \rho_{H_2}((\theta(x)\theta(y)) = \rho_{H_2}((\theta(x)\theta(y))$$

and the result follows accordingly. □

Theorem 3.4 *Let $G \overset{\theta}{\sim} H$, and $\phi : \gamma_G \longrightarrow \rho_H$ be a bigroup homomorphism, then $\gamma_G(\overset{\theta,\phi}{\sim})\rho_H$ if $\phi\gamma_G(x) = \rho_H\theta(x)$.*

Proof Suppose that $\phi\gamma_G(x) = \rho_H\phi(x)$, then $\phi\gamma_{G_1}(x) = \rho_{H_1}\theta(x)$ and $\phi\gamma_{G_2}(x) = \rho_{H_2}\theta(x)$. Since, $\theta : G \to H$ is a bigroup homomorphism, then,

$$\phi\gamma_{G_1}(xy) = \rho_{H_1}\theta(xy) = \rho_{H_1}(\theta(x)\theta(y)), \quad \phi\gamma_{G_2}(xy) = \rho_{H_2}\theta(xy) = \rho_{H_2}(\theta(x)\theta(y)).$$

Hence,

$$\begin{aligned} \phi\gamma_G(xy) &= \phi(\gamma_{G_1 \cup G_2}(xy)) = \max\{\phi(\gamma_{G_1}(xy)), \phi(\gamma_{G_2}(xy))\} \\ &= \max\{\rho_{H_1}(\theta(xy), \rho_{H_2}(\theta(xy)\} = \max\{\rho_{H_1}(\theta(xy), \rho_{H_2}(\theta(xy)\} \\ &= \max\{\rho_{H_1}(\theta(x)\theta(y), \rho_{H_2}(\theta(x)\theta(y),\} = \rho_{H_1 \cup H_2}(\theta(x)\theta(y)) = \rho_H(\theta(x)\theta(y)). \end{aligned}$$

Hence, the proof. □

Definition 3.5 *Let θ be a bigroup homomorphism (isomorphism) from G onto H, let γ_G and ρ_H fuzzy bigroups of G and H respectively. If $\theta\gamma_G = \rho_H$, then we say that γ_G and ρ_H are $I-$homomorphic (isomorphic).*

Definition 3.6 *Let γ_G, ρ_H fuzzy bigroups of the bigroups G and H respectively. If for any $\alpha \in [0,1]$, $(\gamma_G)_\alpha \sim (\rho_H)_\alpha$ $\big((\gamma_G)_\alpha \cong (\rho_H)_\alpha\big)$, then we say γ_G and ρ_H are II-homomorphic(isomorphic).*

Definition 3.7 *Let $(G, +, \cdot)$ and $(H, \oplus, \circ)$ be bigroups with G_1, G_2, and H_1, H_2, as constituting subgroups respectively, and θ an homomorphism from G to H. Let $\gamma_G = \gamma_{G_1} \cup \gamma_{G_2}$ and $\rho_H = \rho_{H_1} \cup \rho_{H_2}$ be fuzzy bigroups of the bigroups G and H respectively. If $\phi : \gamma_G \to \rho_H$ is such that $\phi(\gamma_{G_1}(x)) = \rho_{H_1}(\theta(x))$ and $\phi(\gamma_{G_2}(x)) = \rho_{H_2}(\theta(x))$, then γ_G and ρ_H are said to be $III-$homomorphic. If θ is an isomorphism and ϕ is a monomorphism we call γ_G and ρ_H $III-$isomorphic.*

Definition 3.8 *Let γ_G and ρ_H fuzzy bigroups of the bigroups G and H respectively. Then γ_G and ρ_H are said to be $IV-$homomorphic (isomorphic) if:*

(1) *for any $[\gamma_G]_\alpha$, $\alpha \in [0,1]$, there exists at least one $[\rho_H]_\beta$ such that $[\gamma_G]_\alpha \sim [\rho_H]_\beta$ $\big([\gamma_G]_\alpha \cong [\rho_H]_\beta\big)$, and for any $[\gamma_G]_\eta \supseteq [\gamma_G]_\delta$, there is a $[\rho_H]_\lambda \supseteq [\rho_H]_\mu$ such that $[\gamma_G]_\eta \sim [\rho_H]_\lambda\big([\gamma_G]_\delta \cong \rho_H]_\mu\big).\beta, \eta, \delta, \lambda, \mu \in [0,1]$.*

(2) *for any $[\rho_H]_\beta$, $\beta \in [0,1]$, there exists at least one $[\gamma_G]_\alpha$, such that $[\gamma_G]_\eta \sim [\rho_H]_\beta$ $\big([\gamma_G]_\eta \cong [\rho_H]_\beta\big)$,and for any $[\rho_H]_\lambda \supseteq [\rho_H]_\mu$, there is a $[\gamma_G]_\eta \supseteq [\gamma_G]_\delta$ such that $[\gamma_G]_\eta \sim \rho_H]_\mu\big([\gamma_G]_\eta \cong [\rho_H]_\mu\big)\alpha, \eta, \lambda, \mu, \delta \in [0,1]$.*

Theorem 3.9 *If γ_G and ρ_H are I-homomorphic and γ_G has the sup-property, then they are II-homomorphic.*

Proof Let $\theta : G = (G_1 \cup G_2) \to H = (H_1 \cup H_2)$ be bigroup homomorphism, then, $\theta \mid G_1 : G_1 \to H_1$ is such that $\theta(xy) = \theta(x)\theta(y)$ and $\theta \mid G_2 : G_2 \to H_2$ is such that $\theta(xy) = \theta(x)\theta(y)$ for all $x,\ y \in G_1,\ G_2$. Suppose that $\theta\gamma_G = \rho_H$, we have that $\theta\gamma_{G_1} = \rho_{H_1}$ and $\theta\gamma_{G_2} = \rho_{H_2}$. Now for any $x \in [\gamma_G]_\alpha$, we have that $\gamma_{G_1}(x) \geq \alpha$ and $\gamma_{G_2}(x) \geq \alpha$. [since $[\gamma_G]_\alpha \leq G \implies [\gamma_{G_1}]_\alpha \leq G_1$ and $[\gamma_{G_2}]_\alpha \leq G_2$]. $\rho_{H_1}\big(\theta(x)\big) = sup[\gamma_{G_1}(y)] \geq \gamma_{G_1}(x) \geq \alpha$ so that $\theta(x) \in [\rho_{H_1}]_\alpha$. By similar argument, $\theta(x) \in [\rho_{H_2}]_\alpha$.

For $\forall y \in [\rho_H]_\alpha$, since $[\gamma_G]$ has the sup property, there must be an x in $\theta^{-1}(y)$ such that $\gamma_{G_1}(x) = \rho_{H_1}(x) \geq \alpha$. So x is in $[\gamma_{G_1}]_\alpha$. Similarly, x is in $[\gamma_{G_2}]_\alpha$. Hence if we let $\theta\alpha : [\gamma_{G_1}]_\alpha \longrightarrow [\rho_{H_1}]_\alpha$ and $\theta\alpha : [\gamma_{G_2}]_\alpha \longrightarrow [\rho_{H_2}]_\alpha$ such that $\theta\alpha(x) = \theta(x)$, then $[\gamma_G]_\alpha \sim [\rho_H]_\alpha$. □

Theorem 3.10 *Let γ_G and ρ_H be fuzzy bigroups of the bigroups G and H respectively such that γ_G has the sup-property. Suppose that γ_G and ρ_H are II-homomorphic then they are I-homomorphic if and only if they have a bigroup homomorphism θ from G onto H such that $\theta[\gamma_G]_\alpha = [\rho_H]_\alpha$.*

Proof Suppose that γ_G and ρ_H are II-homomorphic, it follows that $[\gamma_G]_\circ = [\rho_H]_\circ$ so that $G \sim H$. Let θ be a bigroup homomorphism from G onto H. If $[\gamma_G]$ and $[\rho_H]$ are I homomorphic, then $[\rho_H] = \theta(\gamma_G)$. For any $\alpha \in [0,1]$, if $\theta[\gamma_G]_\alpha \neq [\rho_H]_\alpha$, then there must be a z in $[\rho_H]_\alpha$ and

$[\gamma_G]_\alpha \bigcap \theta^{-1}(z) = \phi$. But $[\rho_H]_\alpha(z) = Sup[\gamma_G]_{(x)=z}(x) \geq \alpha$. Since $[\gamma_G]_\alpha$ has Sup-property, there must be an x^1 in $\theta^{-1}(z)$ such that $\gamma_G(x^1) \geq \alpha$, so $x^1 \in [\gamma_G]_\alpha$ and hence $[\gamma_G]_\alpha \bigcap \theta^{-1}(z) \neq \phi$. Hence, we have a contradiction. Therefore, $\theta[\gamma_G]_\alpha = [\rho_H]_\alpha$.

If $\theta[\gamma_G] = \rho_H$ holds for any $\alpha \in [0,1]$, we prove $\rho_H\theta(x) = \underset{y\in\theta^{-1}\theta(y)}{sup}(y)$. For any $y_o \in H$, let $\alpha_o = [\rho_H]_{(y_o)}$. We know that $\theta[\gamma_G] = \rho_H$. Hence, $\rho_H\theta(x) = \underset{y\in\theta^{-1}\theta(y)}{sup}\gamma_G(y)$. If $\rho_H(y_o) < \underset{\theta(x)=y_o}{sup}\gamma_G(x)$, there must be an $x^{'} \in \theta^{-1}(y_o)$ such that $\gamma_G(x') > \rho_H(y_0)$ since γ_G has the sup-property.

Let $\alpha' = \gamma_G(x^{'})$. Then $\theta[\gamma_G]_\alpha = [\rho_H]_\alpha$. But $y_o \in [\rho_H]_\alpha$ if $x^{'} \in [\gamma_G]_\alpha$. there is no y in ρ_H, such that $\theta(x^{'}) = y$. This is a contradiction. Hence, $\rho_H(y_o) = \underset{\theta(x)=y_o}{sup}\gamma_G(x)$ which indicates $\rho_H = \gamma_G(x)$. Hence, the proof. □

References

[1] Akinola L.S., Agboola A.A.A, Permutable and Mutually Permutable Fuzzy Bigroup, *Proceedings of Jangjeon Mathematical Society*, 13 (1), 2010, pp 98-109.

[2] R.Bělohlávek, V.Vychodil, Algebras with fuzzy equalities, *Fuzzy Sets and Systems*, 157 (2006), pp 161-201.

[3] Chakraborty A. B., Khare S.S., Fuzzy homomorphism and f-fuzzy subgroups generated by a fuzzy subset, *Fuzzy Sets and Systems* 74(1995), pp 259-268.

[4] Jelana I., Miroslav C., Stojan B., Fuzzy homomorphisms of algebras, *Fuzzy Sets and Systems*, 160(16), 2009, pp 2345-2365.

[5] Maggu P.L., On introduction of bigroup concept with its application in industry, *Pure Appl. Math Sci.*, 39(1994), pp171-173.

[6] Maggu P.L. and Rajeev K., On sub-bigroup and its applications, *Pure Appl. Math Sci.*,43 (1996), pp 85-88.

[7] Meiyappan D., *Studies on Fuzzy Subgroups,* Ph.D. Thesis, IIT(Madras), June 1998.

[8] A.Rosenfeld, Fuzzy groups, *J. Math. Anal.Appl.*, 35(1971), pp 512-517.

[9] Su-Yun L, De-Gang C., Wen-Xiang G., Hui W., Fuzzy homomorphisms, *Fuzzy Sets and Systems* 79, 1996, pp235-238.

[10] Vasantha W.B.K. and Meiyappan D., Fuzzy symmetric subgroups and conjugate fuzzy subgroups of a group, *J. Fuzzy Math.*, IFMI, 6(1998), pp 905-913.

[11] Vasantha W.B.K., *Bialgebraic Structures and Smarandache Bialgebraic Structures,* American Research Press, Rehoboth, NM, 2003.

[12] Zadeh L.A., Fuzzy sets, *Inform. and Control* 8(1965), pp 338-353.

[13] Zhu N.D., The homomorphism and isomorphism of fuzzy groups, *Fuzzy Math.*, 2(1984), pp 21-27.

Math.Combin.Book Ser. Vol.4(2012), 33-39

$\mathfrak{b}-$Smarandache m_1m_2 Curves of Biharmontic New Type $\mathfrak{b}-$Slant Helices According to Bishop Frame in the Sol Space $\boldsymbol{Sol}^3$

Talat KÖRPINAR and Essin TURHAN

(Department of Mathematics of Fırat University, 23119, Elazığ, TURKEY)

E-mail: talatkorpinar@gmail.com, essin.turhan@gmail.com

Abstract: In this paper, we study $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. We characterize the $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the $\boldsymbol{Sol}^3$.

Key Words: New type $\mathfrak{b}-$slant helix, Sol space, curvatures.

AMS(2010): 53A04, 53A10

§1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be *biharmonic* if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \operatorname{tr}\nabla^\phi d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \operatorname{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1.1}$$

and called the *bitension field* of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organized as follows: Firstly, we study $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant helix in the $\boldsymbol{Sol}^3$. Secondly, we characterize the $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves in terms of their Bishop curvatures. Finally, we find explicit equations of $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves in the $\boldsymbol{Sol}^3$.

§2. Riemannian Structure of Sol Space $\boldsymbol{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\boldsymbol{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

[1]Received August 28, 2012. Accepted December 2, 2012.

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$ [11,12].

Note that the Sol metric can also be written as:

$$g_{\boldsymbol{Sol}^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

Proposition 2.1 *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g_{\boldsymbol{Sol}^3}$, defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

where the (i,j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\boldsymbol{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x,y,z) &\to (x+c, y, z), \\ (x,y,z) &\to (x, y+c, z), \\ (x,y,z) &\to \left(e^{-c}x, e^{c}y, z+c\right). \end{aligned}$$

§3. Biharmonic New Type $\mathfrak{b}-$Slant Helices in Sol Space $\boldsymbol{Sol}^3$

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion [14,15] and

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{n}) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{b},\mathbf{b}) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{n}) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{b}) = g_{\boldsymbol{Sol}^3}(\mathbf{n},\mathbf{b}) = 0. \end{aligned} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [1]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g_{\boldsymbol{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2) = 1, \\ g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_1) &= g_{\boldsymbol{Sol}^3}(\mathbf{t},\mathbf{m}_2) = g_{\boldsymbol{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_2\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \delta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\delta(s), \\ k_2 &= \kappa(s)\sin\delta(s). \end{aligned}$$

The relation matrix may be expressed as

$$\begin{gathered} \mathbf{t} = \mathbf{t}, \\ \mathbf{n} = \cos\delta(s)\,\mathbf{m}_1 + \sin\delta(s)\,\mathbf{m}_2, \\ \mathbf{b} = -\sin\delta(s)\,\mathbf{m}_1 + \cos\delta(s)\,\mathbf{m}_2. \end{gathered}$$

On the other hand, using above equation we have

$$\begin{gathered} \mathbf{t} = \mathbf{t}, \\ \mathbf{m}_1 = \cos\delta(s)\,\mathbf{n} - \sin\delta(s)\,\mathbf{b} \\ \mathbf{m}_2 = \sin\delta(s)\,\mathbf{n} + \cos\delta(s)\,\mathbf{b}. \end{gathered}$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1e_1 + t^2e_2 + t^3e_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{3.5}$$

Theorem 3.1 *$\gamma : I \longrightarrow \boldsymbol{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$\begin{aligned} k_1^2 + k_2^2 &= \text{constant} \neq 0, \\ k_1'' - \left[k_1^2 + k_2^2\right]k_1 &= -k_1\left[2m_2^3 - 1\right] - 2k_2m_1^3m_2^3, \\ k_2'' - \left[k_1^2 + k_2^2\right]k_2 &= 2k_1m_1^3m_2^3 - k_2\left[2m_1^3 - 1\right]. \end{aligned} \tag{3.6}$$

Theorem 3.2 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix with constant slope. Then, the position vector of γ is*

$$\begin{aligned}\gamma(s) &= [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}]\mathbf{e}_1\\ &+[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\mathcal{S}_1\sin[\mathcal{S}_1 s+\mathcal{S}_2]]+\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}]\mathbf{e}_2\\ &+[-\sin\mathcal{M}s+\mathcal{S}_3]\mathbf{e}_3, \qquad (3.7)\end{aligned}$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are constants of integration, [8].

We can use Mathematica in Theorem 3.4, yields

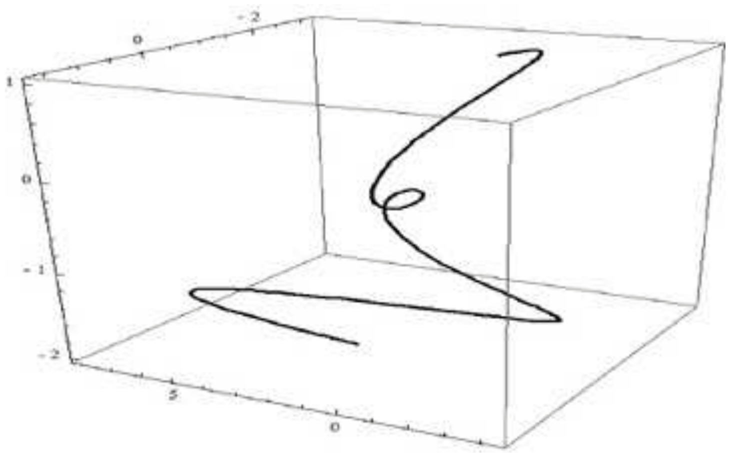


Fig.1

§4. $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ Curves of Biharmonic New Type $\mathfrak{b}-$Slant Helices in $\boldsymbol{Sol}^3$

To separate a Smarandache $\mathbf{m_1 m_2}$ curve according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curve.

Definition 4.1 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix and $\{\mathbf{t},\mathbf{m_1},\mathbf{m_2}\}$ be its moving Bishop frame. $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves are defined by*

$$\gamma_{\mathbf{m_1 m_2}} = \frac{1}{\sqrt{k_1^2+k_2^2}}\left(\mathbf{m_1}+\mathbf{m_2}\right). \qquad (4.1)$$

Theorem 4.2 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the equation of $\mathfrak{b}-$Smarandache $\mathbf{m_1 m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant*

helix is given by

$$\begin{aligned}\gamma_{\mathbf{m_1m_2}}(s) &= \frac{1}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]+\cos[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_1\\ &\quad+\frac{1}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]-\sin[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_2\\ &\quad+\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]\mathbf{e}_3,\end{aligned} \tag{4.2}$$

where $\mathcal{C}_1, \mathcal{C}_2$ are constants of integration.

Proof Assume that γ is a non geodesic biharmonic new type $\mathfrak{b}-$slant helix according to Bishop frame.

From Theorem 3.2, we obtain

$$\mathbf{m}_2=\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1+\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2+\cos\mathcal{M}\mathbf{e}_3, \tag{4.3}$$

where $\mathcal{S}_1, \mathcal{S}_2 \in \mathbb{R}$.

Using Bishop frame, we have

$$\mathbf{m}_1=\cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1-\sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2. \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.1) we have (4.2), which completes the proof. □

In terms of Eqs. (2.1) and (4.2), we may give:

Corollary 4.3 *Let $\gamma : I \longrightarrow \boldsymbol{Sol}^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the parametric equations of $\mathfrak{b}-$Smarandache $\mathbf{tm_1m_2}$ curves of biharmonic new type $\mathfrak{b}-$slant helix are given by*

$$\begin{aligned}x_{\mathbf{tm_1m_2}}(s) &= \frac{e^{-\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]}}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]+\cos[\mathcal{S}_1 s+\mathcal{S}_2]],\\ y_{\mathbf{tm_1m_2}}(s) &= \frac{e^{\frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}]}}{\sqrt{k_1^2+k_2^2}}[\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]-\sin[\mathcal{S}_1 s+\mathcal{S}_2]],\\ z_{\mathbf{tm_1m_2}}(s) &= \frac{1}{\sqrt{k_1^2+k_2^2}}[\cos\mathcal{M}],\end{aligned} \tag{4.5}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are constants of integration.

Proof Substituting (2.1) to (4.2), we have (4.5) as desired. □

We may use Mathematica in Corollary 4.3, yields

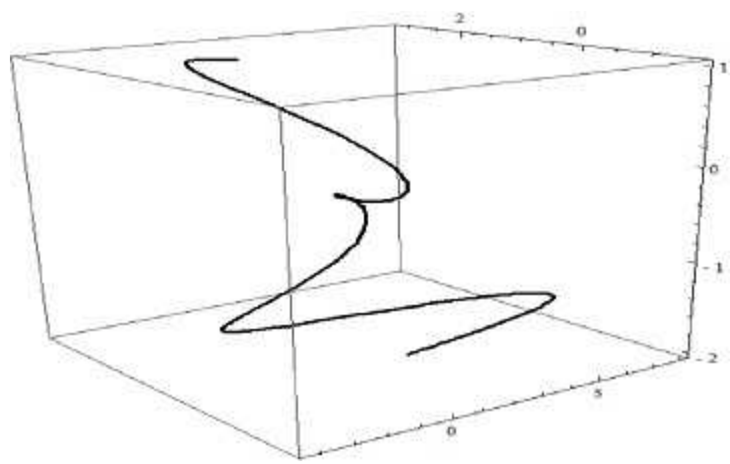


Fig.2

References

[1] L.R.Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly*, 82 (3) (1975) 246-251.

[2] R.M.C.Bodduluri, B.Ravani, Geometric design and fabrication of developable surfaces, *ASME Adv. Design Autom.*, 2 (1992), 243-250.

[3] F.Dillen, W.Kuhnel, Ruled Weingarten surfaces in Minkowski 3-space, *Manuscripta Math.* 98 (1999), 307–320.

[4] I.Dimitric, Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector, *Bull. Inst. Math. Acad. Sinica* 20 (1992), 53–65.

[5] J.Eells and L.Lemaire, A report on harmonic maps, *Bull. London Math. Soc.* 10 (1978), 1–68.

[6] J.Eells and J.H.Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.* 86 (1964), 109–160.

[7] G.Y.Jiang, 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math. Ser.* A 7(2) (1986), 130–144.

[8] T.Körpınar and E.Turhan, Biharmonic new type b-slant helices according to Bishop frame in the sol space, (submitted).

[9] M.A.Lancret, Memoire sur les courbes ‘a double courbure, *Memoires presentes alInstitut* 1 (1806), 416-454.

[10] E.Loubeau and S.Montaldo, Biminimal immersions in space forms,preprint, 2004, *math.DG /0405320* v1.

[11] Y.Ou and Z.Wang, Linear Biharmonic Maps into Sol, Nil and Heisenberg Spaces, *Mediterr. J. Math.* 5 (2008), 379–394

[12] S.Rahmani, Metriqus de Lorentz sur les groupes de Lie unimodulaires, de dimension trois, *Journal of Geometry and Physics*, 9 (1992), 295-302.

[13] D.J.Struik, *Lectures on Classical Differential Geometry,* Dover, New-York, 1988.

[14] E.Turhan and T. Körpınar, On Characterization Of Timelike Horizontal Biharmonic Curves In The Lorentzian Heisenberg Group Heis^3, *Zeitschrift für Naturforschung A- A Journal of Physical Sciences*, 65a (2010), 641-648.

[15] E.Turhan and T.Körpınar, Parametric equations of general helices in the sol space $\mathfrak{Sol}^3$, *Bol. Soc. Paran. Mat.* 31 (1) (2013), 99–104.

Math.Combin.Book Ser. Vol.4(2012), 40-51

On $(r, 2, (r-1)(r-1))$-Regular Graphs

N.R.Santhi Maheswari

(Department of Mathematics, G.Venkataswamy Naidu College, Kovilpatti.)

C.Sekar

(Department of Mathematics, Aditanar College of Arts and Science, Tiruchendur.)

E-mail: nrsmaths@yahoo.com, sekar.acas@gmail.com

Abstract: A graph G is called $(r, 2, k)$-regular if $d(v) = r$ and $d_2(v) = k$ for all v in G. In this paper, we study few properties possessed by $(r, 2, k)$- regular graphs. Further, we discuss in particular the $(r, 2, (r-1)(r-1))$-regular graphs and have given a method to construct $(r, 2, (r-1)(r-1))$-regular graph on $4 \times 2^{r-2}$ vertices.

Key Words: Regular graph, girth, (d, k)-regular, $(2, k)$-regular, distance degree regular, semi regular, diameter.

AMS(2010): 05C12

§1. Introduction

Throughout this paper, by a graph we mean a finite, simple, connected, undirected graph $G(V, E)$. For notations and terminology, we follow [6]. The degree of a vertex v is the number of vertices adjacent to v and it is denoted by $d(v)$. If all the vertices of a graph have the same degree r, we call that graph r-regular.

Distance-degree regular graphs by G.S. Bloom, J.K.Kennedy and L.V.Quintas [3] suggests another way to look at regular graphs. In order to consider another approach to define regular graph, they use the idea of distance. For a connected graph G, the distance between two vertices u and v is the length of the shortest (u, v)-path.

In any graph G, $d(u, v) = 1$ if and only if u and v are adjacent. Therefore, the degree of a vertex v is the number of vertices at a distance 1 from v, and $d_d(v)$ is defined as the number of vertices at a distance d from v. Hence $d_1(v) = d(v)$ and $N_d(v)$ denote the set of all vertices that are at a distance d away from v in a graph G. Hence $N_1(v) = N(v)$.

A graph G is called distance d-regular if every vertex of G has the same number of vertices at a distance d from it [5]. Let us call a graph (d, k)-regular if every vertex of G has exactly k vertices at a distance d from it. Regular graph is one in which each vertex is at a distance one away from exactly the same number of vertices. The $(1, r)$-regular graphs are nothing but our usual r-regular graphs.

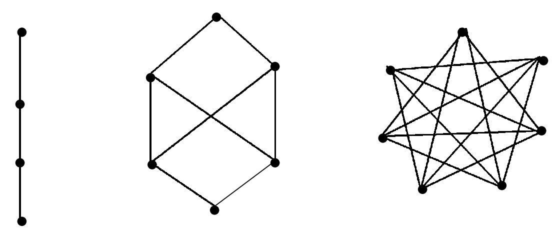
Just as a regular graph, we define a $(2, k)$-regular graph to be a graph in which each vertex

[1]Received August 21, 2012. Accepted December 5, 2012.

is at a distance two away from exactly k vertices. We denote by $d_2(v)$, the number of vertices at a distance two away from v.

This $(2, k)$-regular graph has been introduced in the name of k-semi regular graphs by Alison Northup [2]. A graph G is called semi regular if each vertex in the graph is at a distance two away from exactly the same number of vertices. If each vertex is at a distance two away from n other vertices, we call that graph n-semi regular. Girth of a graph is the smallest cycle in that graph and diameter of graph G is $\max\{d(u, v)/u, v \text{ in } G\}$.

Note that $(2, k)$-regular graphs may be regular or may not be. For example, in Figure 1, the graphs (a) and (b) are non-regular graphs, where as (c) is a regular graph. But all of them are $(2, k)$-regular.



(2, 1) - regular (2, 2)- regular (2, 2) - regular
(a) (b) (c)

Figure 1

A graph is said to be $(r, 2, k)$-regular if $d(v) = r$ and $d_2(v) = k$, for all v in G. If G is $(r, 2, k)$-regular graph, then $0 \leq k \leq r(r-1)$ [8]. Then there arise a question that "*Is it possible to construct the graphs for all values of k lies between* 0 *and* $r(r-1)$, *for any* r *?* " With this motivation, we have constructed $(r, 2, k)$-regular graph for $k = r(r-1)$ in [8]. In this paper, we try it for the case $k = (r-1)(r-1)$ and have given a method to construct $(r, 2, (r-1)(r-1))$-regular graph on $4 \times 2^{r-2}$ vertices.

§2. $(r, 2, k)$-Regular Graphs

Definition 2.1 *A graph is $(r, 2, k)$ - regular if each vertex of G is at a distance one from exactly r vertices and each vertex of G is at a distance two away from exactly k vertices of G. That is, $d(v) = r$ and $d_2(v) = k$, for all vertices v in G.*

Example 2.2 Some $(r, 2, k)$-regular graphs for $0 \leq k \leq r(r-1)$.

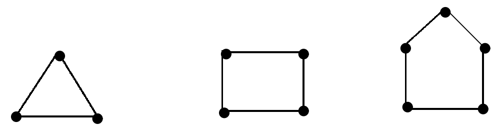
(i) $r = 0$, $k = 0$.

$(0, 2, 0)$-regular

(ii) $r = 1$, $k = 0$.

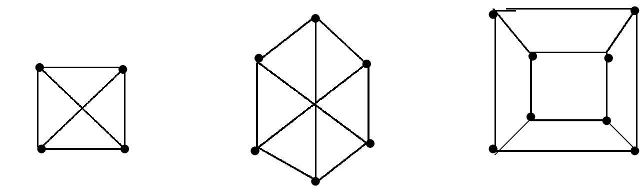
$(1, 2, 0)$-regular.

(*iii*) $r = 2$, k lies between 0 and 2.

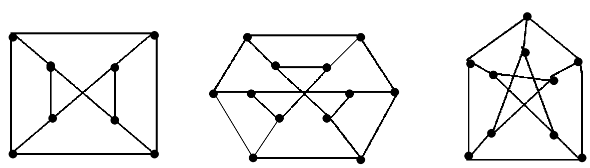


(2,2,0)-regular (2,2,1)-regular (2,2,2)-regular

(*iv*) $r = 3$, k lies between 0 and 6.



(3,2,0)-regular (3,2,2)-regular (3,2,3)-regular



(3,2,4)-regular (3,2,5)-regular (3,2,6)-regular

Figure 2

But, there is no $(3, 2, 1)$-regular graph. Now the question arises in our mind, *is there exists an odd regular with* $(2, 1)$ *regular graph?*. The answer is "NO".

Theorem 2.3 *For any odd* $r \geq 3$*, there is no* $(r, 2, 1)$*-regular graph.*

Proof Let G be an odd regular graph. Suppose G is (2,1)-regular graph, then G is of type P_4 or $\overline{UP_2}$ [2]. Since G is a regular graph, G must be of type $\overline{UP_2}$. Therefore, G is a regular graph of even degree, which is a contradiction. Therefore G is not $(2, 1)$-regular. Hence, there is no $(2, 1)$-regular graph for $r = 3, 5, 7, 9 \cdots$. □

Theorem 2.4 *Any* $(r, 2, k)$*-regular graph has at least* $k + r + 1$ *vertices.*

Proof Let G be a $(r, 2, k)$-regular graph. Then each vertex v is adjacent with r-vertices and non-adjacent with at least k-vertices. Therefore, G has at least $k + r + 1$ vertices. □

Theorem 2.5 *Any* $(2, k)$*-regular graph with* max *deg* r *has at least* $k + r + 1$ *vertices.*

Theorem 2.6 *If* r *and* k *are odd, then* $(r, 2, k)$*-regular graph has at least* $k + r + 2$ *vertices.*

Proof If r and k are odd, then there is no $(r, 2, k)$-regular graph of order $r + k + 1$, since odd regular graphs have only even number of vertices. □

Theorem 2.7 *For any $r \geq 2$ and $k \geq 1$, G is a $(r, 2, k)$-regular graph of order $r + k + 1$ if and only if $diam(G) = 2$.*

Proof Suppose G is a $(r, 2, k)$-regular graph with $r+k+1$ vertices such that $r \geq 2$ and $k \geq 1$. Let v be any vertex of G and v is adjacent to r-vertices $v_1, v_2, v_3, \ldots, v_r$. That is $d(v, v_i) = 1$, for $i = 1, 2, 3, \cdots, r$. Also v is at a distance two away from exactly k vertices $u_1, u_2, u_3, \cdots, u_k$. That is $d(v, u_i) = 2$, for $i = 1, 2, 3, \ldots, k$. Therefore , $diam(G) = \max\{d(u, v)|u, v \text{ in } G\} = 2$.

Conversely, let $diam(G) = 2$ and G be such a r-regular with n vertices. Let v be any vertex in $V(G)$ and $d(v) = r$, for all v in G. That is, v is adjacent with r vertices and non adjacent with $(n-r-1)$ vertices. Since $diam(G) = 2$, then remaining $n-r-1$ vertices are at a distance two away from v. Therefore, G is a $(r, 2, n-r-1)$-regular graph of order $n = r+(n-r-1)+1$ vertices. □

§3. $(r, 2, (r-1)(r-1))$-Regular Graphs

In this section, we have given a method to construct a $(r, 2, (r-1)(r-1))$-regular graph with $4 \times 2^{r-2}$ vertices, for any $r > 1$.

Definition 3.1 *A graph G is said to be $(r, 2, (r-1)(r-1))$-regular if every vertex has degree r and every vertex in the graph G is at a distance two away from exactly $(r-1)(r-1)$ number of vertices. That is, a graph G is $(r, 2, (r-1)(r-1))$-regular if $d(v) = r$ and $d_2(v) = (r-1)(r-1)$, for all v in G.*

Example 3.2 The following examples are clearly $(r, 2, (r-1)(r-1))$-regular graphs.

1. K_2 is a $(1, 2, 0)$-regular graph.
2. C_4 is a $(2, 2, (2-1)(2-1))$-regular graph.
3. Graphs shown in Figure 3 are $(3, 2, (3-1)(3-1))$-regular graphs.

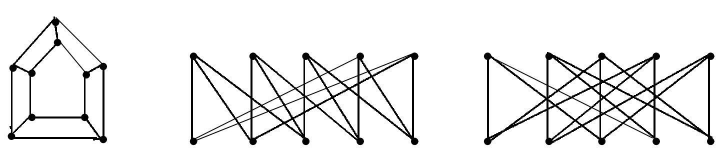


Figure 3.

Theorem 3.3 *For $r > 1$, if G is a $(r, 2, (r-1)(r-1))$-regular graph, then G has girth four.*

Proof Let G be a $(r, 2, (r-1)(r-1))$-regular graph and v be any vertex of G. Let $N(v) = \{v_1, v_2, v_3, \cdots, v_r\}$. Then $d(v) = r$ and $d_2(v) = (r-1)(r-1)$, for all v in G. At least one vertex of $N_2(v)$ is adjacent with more than one vertex of $N(v)$ and $G(N(v))$ has no edges. Therefore, G does not contains triangles and G contains four cycle. Therefore G has girth four. □

Next, we see the main result of this paper.

Theorem 3.4 *For any $r \geq 1$, there is a $(r, 2, (r-1)(r-1))$-regular graph on $4 \times 2^{r-2}$ vertices.*

Proof When $r = 1$, K_2 is the required graph. When $r = 2$, C_4 is the required graph. Let us prove this result by induction on r. Let G be a graph with vertex set $V(G) = \{x_i^{(1)}, x_i^{(2)} | 0 \leq i \leq 3\}$ and edge set $E(G) = \{x_0^{(1)}x_2^{(1)}, x_0^{(1)}x_3^{(1)}, x_1^{(1)}x_2^{(1)}, x_1^{(1)}x_3^{(1)}\} \cup \{x_i^{(1)}x_i^{(2)}, x_i^{(2)}x_{i+1}^{(2)} | 0 \leq i \leq 3\}$ (Subscripts are taken modulo 4). Figure 4 represents the graph 3-regular graph G.

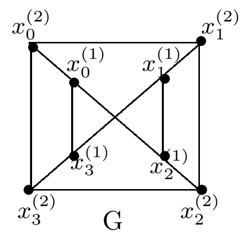


Figure 4

$$N_2(x_i^{(1)}) = \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \text{ and } d_2(x_i^{(1)}) = 4 = (3-1)(3-1).$$
$$N_2(x_i^{(2)}) = \{x_{i+2}^{(2)}, x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+3}^{(1)}\} \text{ and } d_2(x_i^{(2)}) = 4 = (3-1)(3-1).$$

G is a $(3, 2, (3-1)(3-1))$-regular graph on $4 \times 2^{3-2} = 8$ vertices.

Step 1 Take another copy of G as G'. $V(G') = \{x_i^{(3)}, x_i^{(4)} | 0 \leq i \leq 3\}$ and $E(G') = \{x_0^{(3)}x_2^{(3)}, x_0^{(3)}x_3^{(3)}, x_1^{(3)}x_2^{(3)}, x_1^{(3)}x_3^{(3)}\} \cup \{x_i^{(3)}x_i^{(4)}, x_i^{(4)}x_{i+1}^{(4)} | 0 \leq i \leq 3\}$ (Subscripts are taken modulo 4).

The desired graph G_1 has the vertex set $V(G_1) = V(G) \cup V(G')$ and edge set $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}x_{i+1}^{(4)}, x_i^{(2)}x_i^{(3)} | 0 \leq i \leq 3\}$ (Subscripts are taken modulo 4). Now the resulting graph G_1 is 4 regular graph having $4 \times 2^{4-2} = 16$ vertices. Figure 5 represents the graph G_1.

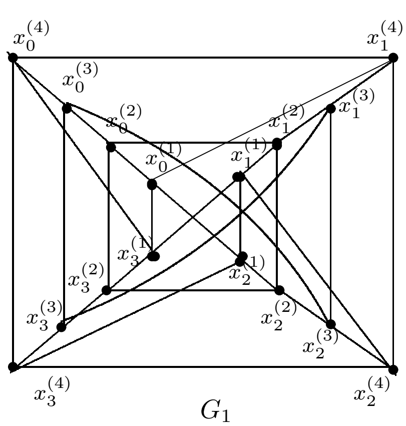


Figure 5.

Next consider the edges $x_i^{(1)} x_{i+1}^{(4)}$ for integers $0 \le i \le 3$. For $(0 \le i \le 3)$,

$$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}\} \text{ in } G \text{ and } |N(x_i^{(1)})| = 3 \text{ in } G \ (0 \le i \le 3).$$
$$N(N(x_i^{(1)})) = \{x_{i+3}^{(4)}, x_i^{(4)}, x_i^{(3)}\} \text{ in } G' \text{ and } |N(x_i^{(1)})| = 3 \text{ in } G' \ (0 \le i \le 3).$$
$$N(x_{i+1}^{(4)}) = \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G' \text{ and } |N(x_{i+1}^{(4)})| = 3 \text{ in } G' \ (0 \le i \le 3).$$
$$N(N(x_{i+1}^{(4)})) = \{x_{i+3}^{(1)}, x_{i+1}^{(1)}, x_{i+1}^{(2)}\} \text{ in } G \text{ and } |N(N(x_{i+1}^{(4)}))| = 3 \text{ in } G \ (0 \le i \le 3)$$

The discussion is divided into the following cases.

(1) To find d_2 of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)} | 0 \le i \le 3\}$ in G_1.

$$\begin{aligned} N_2(x_i^{(1)}) \text{ in } G_1 &= N_2(x_i^{(1)}) \text{ in } G \ \cup N(x_{i+1}^{(4)}) \text{ in } G' \cup N(N(x_i^{(1)})) \text{ in } G' \\ &= \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \text{ in } G \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G' \\ &\cup \{x_{i+3}^{(4)}, x_i^{(4)}, x_i^{(3)}\} \text{ in } G' \\ &= \{x_{i+1}^{(1)}, x_{i+1}^{(2)}, x_{i+2}^{(2)}, x_{i+3}^{(2)}\} \cup \{x_i^{(4)}, x_{i+2}^{(4)}, x_{i+1}^{(3)}, x_{i+3}^{(4)}, x_i^{(3)}\} \text{ in } G' \end{aligned}$$

Here $x_i^{(4)}$ is the common element in $N(x_{i+1}^{(4)})$ in G' and $N(N(x_i^{(1)}))$ in G'.

$$\begin{aligned} d_2(x_i^{(1)}) \text{ in } G_1 &= d_2(x_i^{(1)}) \ in \ G + (|N(x_{i+1}^{(4)})| \text{ in } G' + |N(N(x_i^{(1)}))| \text{ in } G') - 1. \\ &= 4 + (3+3) - 1 = 9 = (4-1)(4-1) \ (0 \le i \le 3). \end{aligned}$$

(2) To find d_2 of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)} | 0 \le i \le 3\}$ in G_1.

$$\begin{aligned} N_2(x_{i+1}^{(4)}) \text{ in } G_1 &= N_2(x_{i+1}^{(4)}) \text{ in } G' \ \cup N(x_i^{(1)}) \text{ in } G \ \cup N(N(x_{i+1}^{(4)})) \text{ in } G \\ &= \{x_{i+3}^{(4)}, x_{i+3}^{(3)}, x_{i+2}^{(3)}, x_i^{(3)}\} \cup \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}\} \text{ in } G \\ &\cup \{x_{i+3}^{(1)}, x_{i+1}^{(1)}, x_{i+1}^{(2)}\} \text{ in } G \\ &= \{x_{i+3}^{(4)}, x_{i+3}^{(3)}, x_{i+2}^{(3)}, x_i^{(3)}\} \cup \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}, x_{i+1}^{(1)}, x_{i+1}^{(2)}\} \text{ in } G. \end{aligned}$$

Here, $x_{i+3}^{(1)}$ is the common element in $N(x_i^{(1)})$ in G and $N(N(x_{i+1}^{(4)}))$ in G.

$$\begin{aligned} d_2(x_{i+1}^{(4)}) \text{ in } G_1 &= d_2(x_{i+1}^{(4)}) \text{ in } G' + (|(N(x_i^{(1)})| \text{ in } G + |N(N(x_{i+1}^{(4)}))| \text{ in } G - 1 \\ &= 6 + (3+3) - 1 = 9 = (4-1)(4-1) \ (0 \le i \le 3). \end{aligned}$$

Next consider the edges $x_i^{(2)} x_i^{(3)}$ for integers $0 \le i \le 3$.

$$N(x_i^{(2)}) = \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(1)}\} \text{ in G and } |N(x_i^{(2)})| = 3 \text{ in } G, \ 0 \le i \le 3;$$
$$N(N(x_i^{(2)})) = \{x_{i+1}^{(3)}, x_{i+3}^{(3)}, x_{i+1}^{(4)}\} \text{ in G}' \text{ and } |N(N(x_i^{(2)}))| = 3 \text{ in } G', \ 0 \le i \le 3;$$
$$N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}\} \text{ in G}' \text{ and } |N(x_i^{(3)})| = 3 \text{ in } G', \ 0 \le i \le 3;$$
$$N(N(x_i^{(3)})) = \{x_{i+2}^{(2)}, x_{i+3}^{(2)}, x_{i+3}^{(1)}\} \text{ in G and } |N(N(x_i^{(3)}))| = 3 \text{ in } G, \ 0 \le i \le 3).$$

(3) To find d_2 of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x_i^{(2)}|0 \leq i \leq 3\}$ in G_1.

$$\begin{aligned}N_2(x_i^{(2)}) \text{ in } G_1 &= N_2(x_i^{(2)}) \text{ in } G \ \cup N(x_i^{(3)}) \text{ in } G' \ \cup N(N(x_i^{(2)})) \text{ in } G' \\ &= \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+2}^{(2)}, x_{i+3}^{(1)}\} \text{ in } G \ \cup \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}\} \text{ in } G' \\ &\quad \cup \{x_{i+1}^{(3)}, x_{i+3}^{(3)}, x_{i+1}^{(4)}\} \text{ in } G' \\ &= \{x_{i+1}^{(1)}, x_{i+2}^{(1)}, x_{i+2}^{(2)}, x_{i+3}^{(1)}\} \text{ in } G \ \cup \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}\} \text{ in } G'.\end{aligned}$$

Here, $x_{i+3}^{(3)}$ is the common element in $N(x_i^{(3)})$ in G' and $N(N(x_i^{(2)}))$ in G',

$$\begin{aligned}d_2(x_i^{(2)}) \text{ in } G_1 &= d_2(x_i^{(2)}) \text{ in } G + (|(N(x_i^{(3)})| \text{ in } G' + |N(N(x_i^{(2)}))| \text{ in} G') - 1 \\ &= 4 + (3+3) - 1 = 9 = (4-1)(4-1) \ (0 \leq i \leq 3).\end{aligned}$$

(4) To find d_2 of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)}|0 \leq i \leq 3\}$ in G_1.

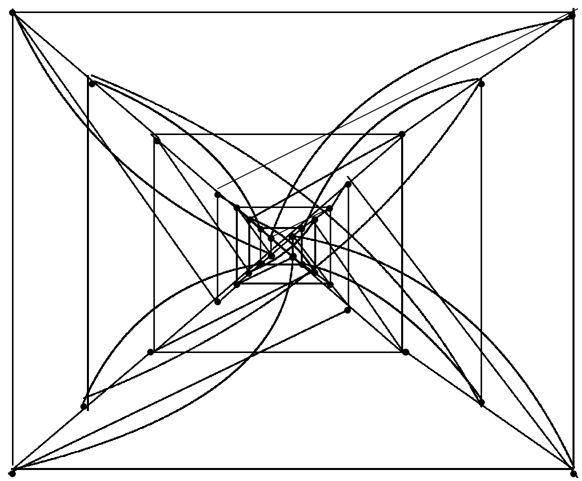
$$\begin{aligned}N_2(x_i^{(3)}) \text{ in } G_1 &= N_2(x_i^{(3)}) \text{ in } G' \ \cup N(x_i^{(2)}) \text{ in } G \ \cup N(N(x_i^{(3)})) \text{ in } G \\ &= \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G' \ \cup \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(1)}\} \text{ in } G \\ &\quad \cup \{x_{i+2}^{(2)}, x_{i+3}^{(2)}, x_{i+3}^{(1)}\} \text{ in } G \\ &= \{x_{i+1}^{(4)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_{i+1}^{(3)}\} \text{ in } G' \ \cup \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(1)}, x_{i+2}^{(2)}, x_{i+3}^{(1)}\} \text{ in } G.\end{aligned}$$

Here, $x_{i+3}^{(2)}$ is the common element in $N(x_i^{(2)})$ in G and $N(N(x_i^{(3)}))$ in G.

$$\begin{aligned}d_2(x_i^{(3)}) \text{ in } G_1 &= d_2(x_i^{(3)}) \text{ in } G' + (|(N(x_i^{(2)})| \text{ in } G + \text{in } |N(N(x_i^{(3)}))| \text{ in } G) - 1. \\ &= 4 + (3+3) - 1 = 9 = (4-1)(4-1) \ (0 \leq i \leq 3).\end{aligned}$$

In G_1 for integers $1 \leq t \leq 4$, $d_2(x_i^{(t)}) = (4-1)(4-1)$ for integers $0 \leq i \leq 3$. Thus G_1 is $(4, 2, (4-1)(4-1))$-regular graph on $4 \times 2^{4-2} = 16$ vertices with the vertex set $V(G_1) = \{x_i^{(t)}|1 \leq t \leq 2^{4-2}, 0 \leq i \leq 3\}$ and $E(G_1) = E(G) \cup E(G') \cup \{x_i^{(1)}x_{i+1}^{(4)}, x_i^{(2)}x_i^{(3)}|0 \leq i \leq 3\}$. Therefore, the result is true for $r = 4$.

Step 2 Take another copy of G_1 as G_1' with the vertex set. $V(G_1') = \{x_i^{(t)}|2^{4-2}+1 \leq t \leq 2^{4-1}, 0 \leq i \leq 3\}$ and each $x_i^{(t)}, (2^{4-2}+1 \leq t \leq 2^{4-1})$, corresponds to $x_i^{(t)}, 1 \leq t \leq 2^{4-2}$ for $(\leq i \leq 3$. The desired graph G_2 has the vertex set $V(G_2) = V(G_1) \cup V(G_1')$ and edge set $E(G_2) = E(G_1) \cup E(G_1') \cup \{x_i^{(1)}x_{i+1}^{(8)}, x_i^{(2)}x_i^{(7)}, x_i^{(3)}x_{i+1}^{(6)}, x_i^{(4)}x_i^{(5)}\}$. Now the resulting graph G_2 is 5 regular graph having $4 \times 2^{5-2} = 32$ vertices. Figure 6 represents the graph G_2.



G_2

Figure 6.

Consider the edges $x_i^{(1)} x_{i+1}^{(8)}$ for $0 \leq i \leq 3$

$$N(x_i^{(1)}) = \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(x_i^{(1)})| = 4 \text{ in } G_1$$
$$N(N(x_i^{(1)})) = \{x_{i+3}^{(8)}, x_i^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(N(x_i^{(1)}))| = 4 \text{ in } G_1'$$
$$N(x_{i+1}^{(8)}) = \{x_i^{(8)}, x_{i+2}^{(8)}, x_{i+1}^{(7)}, x_i^{(5)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(x_{i+1}^{(8)})| = 4 \text{ in } G_1'$$
$$N(N(x_{i+1}^{(8)})) = \{x_{i+1}^{(1)}, x_{i+3}^{(1)}, x_{i+1}^{(2)}, x_i^{(4)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(N(x_{i+1}^{(8)}))| = 4 \text{ in } G_1.$$

(1) To find d_2 of each vertex in $C^{(1)}$, where $C^{(1)}$ is the cycle induced by the vertices $\{x_i^{(1)} | 0 \leq i \leq 3\}$ in G_2.

$$\begin{aligned} N_2(x_i^{(1)}) \text{ in } G_2 &= N_2(x_i^{(1)}) \text{ in } G_1 \cup N(x_{i+1}^{(8)}) \text{ in } G_1' \cup N(N(x_i^{(1)})) \text{ in } G_1' \\ &= N_2(x_i^{(1)}) \text{ in } G_1 \cup \{x_i^{(8)}, x_{i+2}^{(8)}, x_{i+1}^{(7)}, x_i^{(5)}\} \text{ in } G_1' \\ &\quad \cup \{x_{i+3}^{(8)}, x_i^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1' \\ &= N_2(x_i^{(1)}) \text{ in } G_1 \cup \{x_i^{(8)}, x_{i+2}^{(8)}, x_{i+1}^{(7)}, x_i^{(5)}, x_{i+3}^{(8)}, x_i^{(7)}, x_{i+1}^{(5)}\} \text{ in } G_1'. \end{aligned}$$

Here $x_i^{(8)}$ is the common element in $N(x_{i+1}^{(8)})$ in G_1' and $N(N(x_i^{(1)}))$ in G_1',

$$\begin{aligned} d_2(x_i^{(1)}) \text{ in } G_1 &= d_2(x_i^{(1)}) \text{ in } G_1 + (|(N(x_{i+1}^{(8)})| \text{ in } G_1' + |N(N(x_i^{(1)}))| \text{ in } G_1') - 1. \\ &= 9 + (4+4) - 1 = 16 = (5-1)(5-1) \ (0 \leq i \leq 3). \end{aligned}$$

(2) To find d_2 of each vertex in $C^{(8)}$, where $C^{(8)}$ is the cycle induced by the vertices $\{x_i^{(1)} | 0 \leq i \leq 3\}$ in G_2.

$$\begin{aligned} N_2(x_{i+1}^{(8)}) \text{ in } G_2 &= N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup N(x_i^{(1)}) \text{ in } G_1 \cup N(N(x_{i+1}^{(8)})) \text{ in } G_1. \\ &= N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}, x_{i+1}^{(4)}\} \text{ in } G_1 \\ &\quad \cup \{x_{i+1}^{(2)}, x_{i+1}^{(1)}, x_{i+3}^{(1)}, x_i^{(4)}\} \text{ in } G_1 \\ &= N_2(x_{i+1}^{(8)}) \text{ in } G_1' \cup \{x_{i+2}^{(1)}, x_{i+3}^{(1)}, x_i^{(2)}, x_{i+1}^{(4)} x_{i+1}^{(2)}, x_{i+1}^{(1)}, x_i^{(4)}\} \text{ in } G_1. \end{aligned}$$

Here, $x_{i+3}^{(1)}$ is the common element in $N(x_i^{(1)})$ in G_1 and $N(N(x_{i+1}^{(8)}))$ in G_1,

$$d_2(x_{i+1}^{(8)}) \text{ in } G_2 = d_2(x_{i+1}^{(8)}) \text{ in } G_1' + (|(N(x_i^{(1)})| \text{ in } G_1 + |N(N(x_{i+1}^{(8)}))| \text{ in } G_1) - 1$$
$$d_2(x_{i+1}^{(8)}) \text{ in } G_2 = 9+4+4-1 = 9+8-1 = (5-1)(5-1)\ (0 \le i \le 3).$$

Next consider the edge $x_i^{(2)}x_i^{(7)}$ for integers $0 \le i \le 3$.

$$N(x_i^{(2)}) = \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(1)}, x_i^{(3)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(x_i^{(2)})| = 4 \text{ in } G_1,$$
$$N(N(x_i^{(2)})) = \{x_{i+1}^{(7)}, x_{i+3}^{(7)}, x_{i+1}^{(8)}, x_{i+1}^{(6)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(N(x_i^{(2)}))| = 4 \text{ in } G_1',$$
$$N(x_i^{(7)}) = \{x_{i+2}^{(7)}), x_{i+3}^{(7)}, x_i^{(8)}, x_i^{(6)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(x_i^{(7)})| = 4 \text{ in } G_1',$$
$$N(N(x_i^{(7)})) = \{x_{i+2}^{(2)}, x_{i+3}^{(2)}, x_{i+3}^{(1)}, x_{i+3}^{(3)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(N(x_i^{(7)}))| = 4 \text{ in } G_1.$$

(3) To find d_2 of each vertex in $C^{(2)}$, where $C^{(2)}$ is the cycle induced by the vertices $\{x_i^{(2)} | 0 \le i \le 3\}$ in G_2.

$$\begin{aligned} N_2(x_i^{(2)}) \text{ in } G_2 &= N_2(x_i^{(2)}) \text{ in } G_1 \ \cup N(x_i^{(7)}) \text{ in } G_1' \ \cup \ N(N(x_i^{(2)}))\ in\ G_1' \\ &= N_2(x_i^{(2)})\ in\ G_1 \ \cup \{x_{i+3}^{(7)}, x_{i+2}^{(7)}, x_i^{(6)}, x_i^{(8)}\} \text{ in } G_1' \\ &\quad \cup \{x_{i+1}^{(8)}, x_{i+1}^{(6)}, x_{i+1}^{(7)}, x_{i+3}^{(7)}\} \text{ in } G_1' \\ &= N_2(x_i^{(2)})\ in\ G_1 \ \cup \{x_{i+3}^{(7)}, x_{i+2}^{(8)}, x_i^{(6)}, x_i^{(8)} x_{i+1}^{(8)}, x_{i+1}^{(6)}, x_{i+1}^{(7)}\} \text{ in } G_1'. \end{aligned}$$

Here, $x_{i+3}^{(7)}$ is the common element in $N(x_i^{(7)})$ in and $N(N(x_i^{(2)}))$ in G_1'

$$\begin{aligned} d_2(x_i^{(2)}) \text{ in } G_2 &= d_2(x_i^{(2)}) \text{ in } G_1 \ + (|(N(x_i^{(7)})| \text{ in } G_1' \ + |N(N(x_i^{(2)}))| \text{ in } G_1') - 1. \\ &= 9 + (4+4) - 1 = 16 = (5-1)(5-1)\ (0 \le i \le 3). \end{aligned}$$

(4)To find d_2 of each vertex in $C^{(7)}$, where $C^{(7)}$ is the cycle induced by the vertices $\{x_i^{(1)} | 0 \le i \le 3\}$ in G_2.

$$\begin{aligned} N_2(x_i^{(7)}) \text{ in } G_2 &= N_2(x_i^{(7)}) \text{ in } G_1 \ \cup N(x_i^{(2)}) \text{ in } G_1 \ \cup N(N(x_i^{(7)})) \text{ in } G_1. \\ &= N_2(x_i^{(7)}) \text{ in } G_1' \ \cup \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(3)}, x_i^{(1)}\} \text{ in } G_1 \\ &\quad \cup \{x_{i+3}^{(2)}, x_{i+3}^{(3)}, x_{i+3}^{(1)}, x_{i+2}^{(2)}\} \text{ in } G_1. \\ &= N_2(x_i^{(7)}) \text{ in } G_1' \ \cup \{x_{i+1}^{(2)}, x_{i+3}^{(2)}, x_i^{(3)}, x_i^{(1)}, x_{i+3}^{(3)}, x_{i+3}^{(1)}, x_{i+2}^{(2)}\} \text{ in } G_1. \end{aligned}$$

Here, $x_{i+3}^{(2)}$ is the common element in $N(x_i^{(2)})$ in G_1 and $N(N(x_i^{(7)}))$ in G_1,

$$d_2(x_i^{(7)}) \text{ in } G_2 = (d_2(x_i^{(7)}) \text{ in } G_1' + (|N(x_i^{(2)})|\ in\ G_1 + |N(N(x_1^{(7)}))| \text{ in } G_1) - 1$$
$$d_2(x_i^{(7)}) \text{ in } G_2 = 9+4+4-1 = 9+8-1 = (5-1)(5-1)\ (0 \le i \le 3).$$

Next consider the edge $x_i^{(3)}x_{i+1}^{(6)}$ for integers $0 \le i \le 3$.

$$N(x_i^{(3)}) = \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(x_i^{(3)})| = 4 \text{ in } G_1.$$
$$N(N(x_i^{(3)})) = \{x_{i+3}^{(6)}, x_i^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(N(x_i^{(3)}))| = 4 \text{ in } G_1'.$$
$$N(x_{i+1}^{(6)}) = \{x_{i+2}^{(6)}, x_i^{(6)}, x_{i+1}^{(5)}, x_{i+1}^{(7)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(x_{i+1}^{(6)})| = 4 \text{ in } G_1'.$$
$$N(N(x_{i+1}^{(6)})) = \{x_{i+3}^{(3)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}, x_{i+1}^{(2)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(N(x_{i+1}^{(8)}))| = 4 \text{ in } G_1.$$

(5) To find d_2 of each vertex in $C^{(3)}$, where $C^{(3)}$ is the cycle induced by the vertices $\{x_i^{(3)} | 0 \le i \le 3\}$ in G_2.

$$\begin{aligned} N_2(x_i^{(3)}) \text{ in } G_2 &= N_2(x_i^{(3)}) \text{ in } G_1 \ \cup N(x_{i+1}^{(6)}) \text{ in } G_1' \ \cup \ N(N(x_i^{(3)})) \text{ in } G_1' \\ &= N_2(x_i^{(3)}) \text{ in } G_1 \ \cup \{x_{i+2}^{(6)}, x_i^{(6)}, x_{i+1}^{(5)}, x_{i+1}^{(7)}\} \text{ in } G_1' \\ &\quad \cup \{x_{i+3}^{(6)}, x_i^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G_1' \\ &= N_2(x_i^{(3)}) \text{ in } G_1 \ \cup \{x_{i+2}^{(6)}, x_i^{(6)}, x_{i+1}^{(5)}, x_{i+1}^{(7)}, x_{i+3}^{(6)}, x_i^{(5)}, x_i^{(7)}\} \text{ in } G_1. \end{aligned}$$

Here, $x_i^{(6)}$ is the common element in $N(x_{i+1}^{(8)})$ in G_1' and $N(N(x_i^{(1)}))$ in G_1',

$$\begin{aligned} d_2(x_i^{(3)}) \text{ in } G_1 \ &= d_2(x_i^{(3)}) \text{ in } G_1 + (|N(x_{i+1}^{(6)})| \text{ in } G_1' + |N(N(x_i^{(3)}))| \text{ in } G_1') - 1. \\ &= 9 + (4+4) - 1 = 16 = (5-1)(5-1) \ (0 \le i \le 3). \end{aligned}$$

(6) To find d_2 of each vertex in $C^{(6)}$, where $C^{(6)}$ is the cycle induced by the vertices $\{x_i^{(6)} | 0 \le i \le 3\}$ in G_2.

$$\begin{aligned} N_2(x_{i+1}^{(6)}) \text{ in } G_2 &= N_2(x_{i+1}^{(6)}) \ \mathit{in} \ G_1 \ \cup N(x_i^{(3)}) \text{ in } G_1 \ \cup N(N(x_{i+1}^{(6)})) \text{ in } G_1. \\ &= N_2(x_{i+1}^{(6)}) \text{ in } G_1' \ \cup \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}\} \text{ in } G_1 \\ &\quad \cup \{x_{i+3}^{(3)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}, x_{i+1}^{(2)}\} \text{ in } G_1 \\ &= N_2(x_{i+1}^{(6)}) \text{ in } G_1' \ \cup \{x_{i+2}^{(3)}, x_{i+3}^{(3)}, x_i^{(4)}, x_i^{(2)}, x_{i+1}^{(3)}, x_{i+1}^{(4)}, x_{i+1}^{(2)}\} \text{ in } G_1. \end{aligned}$$

Here, $x_{i+3}^{(3)}$ is the common element in $N(x_i^{(1)})$ in G_1 and $N(N(x_{i+1}^{(8)}))$ in G_1,

$$\begin{aligned} d_2(x_{i+1}^{(6)}) \text{ in } G_2 &= d_2(x_{i+1}^{(6)}) \text{ in } G_1' + (|N(x_i^{(3)})| \text{ in } G_1 + |N(N(x_{i+1}^{(6)}))| \text{ in } G_1) - 1. \\ d_2(x_{i+1}^{(6)}) \text{ in } G_2 &= 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5-1)(5-1) \ (0 \le i \le 3). \end{aligned}$$

Next consider the edge $x_i^{(4)} x_i^{(5)}$ for integers $1 \le i \le 3$.

$$\begin{aligned} N(x_i^{(4)}) &= \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(x_i^{(4)})| = 4, \text{ in } G_1. \\ N(N(x_i^{(4)})) &= \{x_{i+1}^{(5)}, x_{i+3}^{(5)}, x_{i+1}^{(6)}, x_i^{(8)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(N(x_i^{(4)}))| = 4, \text{ in } G_1'. \\ N(x_i^{(5)}) &= \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \text{ in } \mathrm{G}_1' \text{ and } |N(x_i^{(5)})| = 4, \text{ in } G_1'. \\ N(N(x_i^{(5)})) &= \{x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_{i+3}^{(3)}, x_{i+3}^{(1)}\} \text{ in } \mathrm{G}_1 \text{ and } |N(N(x_i^{(5)}))| = 4, \text{ in } G_1. \end{aligned}$$

(7) To find d_2 of each vertex in $C^{(4)}$, where $C^{(4)}$ is the cycle induced by the vertices $\{x_i^{(4)} | 0 \le i \le 3\}$ in G_2.

$$\begin{aligned} N_2(x_i^{(4)}) \text{ in } G_2 &= N_2(x_i^{(4)}) \text{ in } G_1 \ \cup N(x_i^{(5)}) \text{ in } G_1' \ \cup N(N(x_i^{(4)})) \text{ in } G_1' \\ &= N_2(x_i^{(4)}) \text{ in } G_1 \ \cup \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}\} \text{ in } G_1 \\ &\quad \cup \{x_{i+1}^{(5)}, x_{i+3}^{(5)}, x_{i+1}^{(6)}, x_i^{(8)}\} \text{ in } G_1' \\ &= N_2(x_i^{(2)}) \ \mathit{in} \ G_1 \ \cup \{x_{i+2}^{(5)}, x_{i+3}^{(5)}, x_i^{(6)}, x_{i+1}^{(8)}, x_{i+1}^{(5)}, x_{i+1}^{(6)}, x_i^{(8)}\} \text{ in } G_1'. \end{aligned}$$

Here, $x_{i+3}^{(5)}$ is the common element in $N(x_i^{(5)})$ in G_1' and $N(N(x_i^{(4)}))$ in G_1',

$$\begin{aligned} d_2(x_i^{(4)}) \text{ in } G_2 &= d_2(x_i^{(4)}) \text{ in } G_1 + (|N(x_i^{(5)})| \text{ in } G_1' + |N(N(x_i^{(4)}))| \text{ in } G_1') - 1. \\ &= 9 + (4+4) - 1 = 16 = (5-1)(5-1) \ (0 \le i \le 3). \end{aligned}$$

(8) To find d_2 of each vertex in $C^{(5)}$, where $C^{(5)}$ is the cycle induced by the vertices $\{x_i^{(5)}|0 \le i \le 3\}$ in G_2.

$$\begin{aligned}
N_2(x_i^{(5)}) \text{ in } G_2 &= N_2(x_i^{(5)}) \text{ in } G_1' \ \cup N(x_i^{(4)}) \text{ in } G_1 \ \cup N(N(x_i^{(5)})) \text{ in } G_1\\
&= N_2(x_i^{(5)}) \text{ in } G_1' \ \cup \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}\} \text{ in } G_1\\
&\quad \cup \{x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_{i+3}^{(3)}, x_{i+3}^{(1)}\} \text{ in } G_1.\\
&= N_2(x_i^{(5)}) \text{ in } G_1' \cup \{x_{i+1}^{(4)}, x_{i+3}^{(4)}, x_i^{(3)}, x_{i+3}^{(1)}, x_{i+2}^{(4)}, x_{i+3}^{(4)}, x_{i+3}^{(1)}\} \text{ in } G_1.
\end{aligned}$$

Here, $x_{i+3}^{(4)}$ is the common element in $N(x_i^{(4)})$ in G_1 and $N(N(x_i^{(5)}))$ in G_1,

$$\begin{aligned}
d_2(x_i^{(5)}) \text{ in } G_2 &= d_2(x_i^{(5)}) \text{ in } G_1'\\
&\quad + (|N(x_i^{(4)})| \text{ in } G_1 + |N(N(x_i^{(5)}))| \text{ in } G_1) - 1\\
d_2(x_i^{(5)}) \text{ in } G_2 &= 9 + 4 + 4 - 1 = 9 + 8 - 1 = (5-1)(5-1), (0 \le i \le 3).
\end{aligned}$$

In G_2, for $(1 \le t \le 8)$, $d_2(x_i^{(t)}) = (5-1)(5-1)$, for $(0 \le i \le 3)$. G_2 is a $(5, 2, (5-1)(5-1))$ regular graph on $4 \times 2^{5-2} = 32$ vertices with the vertex set $V(G_2) = \{x_i^{(t)}|1 \le t \le 2^{5-2}, 0 \le i \le 3\}$ and $E(G_2) = E(G_1) \cup E(G_1') \cup \{x_i^{(1)}x_{i+1}^{(8)}, x_i^{(2)}x_i^{(7)}, x_i^{(3)}x_{i+1}^{(6)}, x_i^{(4)}x_i^{(5)}\}$. Therefore, the result is true for $r = 5$.

Let us assume this result is true for $r = m+3$. That is, there exists a $(m+3, 2, (m+2)^2)$-regular graph on $4 \times 2^{m+1}$ vertices with the vertex set $V(G_m) = \{x_i^{(t)}|1 \le t \le 2^{m+1}, 0 \le i \le 3\}$ and

$$E(G_m) = E(G_{m-1}) \cup E(G_{m-1}') \bigcup_{t=1}^{2^m} \{x_i^t x_{i+tmod2}^{2^{m+1}-t+1}|0 \le i \le 3\}.$$

That is, for integers $1 \le t \le 2^{m+1}$, $d_2(x_i^{(t)}) = (m+2)(m+2)$ for $0 \le i \le 3$ and $d(x_i^{(t)}) = m+3$.

Take another copy of G_m as G_m' with the vertex set. $V(G_m') = \{x_i^{(t)}|2^{m+1}+1 \le t \le 2^{m+2}, 0 \le i \le 3\}$ and each $x_i^{(t)}, 2^{m+1}+1 \le t \le 2^{m+2}$ corresponds to $x_i^{(t)}, 1 \le t \le 2^{m+1}$ for integers $0 \le i \le 3$.

The desired graph G_{m+1} has the vertex set $V(G_{m+1}) = V(G_m) \cup V(G_m')$ and edge set $E(G_{m+1}) = E(G_m) \cup E(G_m') \bigcup_{t=1}^{m+1} \{x_i^t x_{i+tmod2}^{2^{m+2}-t+1}|0 \le i \le 3\}$. Now the resulting graph G_{m+1} is $(m+4)$ regular graph having $4 \times 2^{m+2}$ vertices. Consider the edges $\bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} x_{i+t(mod\ 2)}^{2^{m+2}-t+1}/(0 \le i \le 3)\}$.

For integers $1 \le t \le 2^{m+1}, d_2$ of each vertex in $C^{(t)}$, where $C^{(t)}$ is the cycle induced by the vertices $\{x_i^{(t)}|0 \le i \le 3\}$ in G_{m+1}.

$$\begin{aligned}
d_2(x_i^{(t)}) \text{ in } G_{m+1} &= d_2(x_i^{(t)}) \text{ in } G_m + |N(x_{i+t(mod\ 2)}^{2^{m+2}-t+1})| \text{ in } G_m' + |N(N(x_i^{(t)}))| \text{ in } G_m'.\\
&= (m+2)(m+2) + ((m+3) + (m+3)) - 1, \qquad (\text{for } 0 \le i \le 3).\\
&= (m+2)(m+2) + 2m + 5).\\
&= m^2 + 6m + 9 = (m+3)(m+3).
\end{aligned}$$

For integers $2^{m+1}+1 \leq t \leq 2^{m+2}, d_2$ of each vertex in $C^{(2^{m+2}-t+1)}$, where $C^{(2^{m+2}-t+1)}$ is the cycle induced by the vertices $\{x_i^{(2^{m+2}-t+1)} | 0 \leq i \leq 3\}$ in G_{m+1}.

$$\begin{aligned} d_2(x_{i+t(mod\ 2)}^{2^{m+2}-t+1}) \text{ in } G_{m+1} &= d_2(x_{i+t(mod\ 2)}^{2^{m+2}-t+1}) \text{ in } G'_m \\ &\quad + |N(x_i^{(t)})| \text{ in } G_m \ + |N(N(x_{i+t(mod\ 2)}^{2^{m+2}-t+1})| \text{ in } G_m. \\ &= (m+3)(m+2)+(m+3)+(m+3)-1, \qquad (\text{for } 0 \leq i \leq 3). \\ &= (m+3)(m+3). \end{aligned}$$

In G_{m+1} for $1 \leq t \leq 2^{m+2}, d_2(x_i^{(t)}) = (m+3)(m+3)$ for $0 \leq i \leq 3$. That is, there exists a $((m+4), 2, (m+3)(m+3))$-regular graph on $4 \times 2^{m+2}$ vertices with the vertex set $V(G_{m+1}) = \{x_i^{(t)} | 1 \leq t \leq 2^{m+2}, 0 \leq i \leq 3\}$ and $E(G_{m+1}) = E(G_m) \cup E(G'_m) \bigcup_{t=1}^{2^{m+1}} \{x_i^{(t)} x_{i+t(mod\ 2)}^{2^{m+2}-t+1} | 0 \leq i \leq 3\}$. That is, for integers $1 \leq t \leq 2^{m+2}, d_2(x_i^{(t)}) = (m+3)(m+3)$ for $0 \leq i \leq 3$ and $d(x_i^{(t)}) = m+4$. If the result is true for $r = m+3$, then it is true for $r = m+4$. Therefore, the result is true for all $r \geq 2$. □

References

[1] Y.Alavi, G.Chartrand, F.R.K.Chung, P Erdos, R.L.Graham and Ortrud R.Oellermann, Highly irregular graphs, *J. Graph Theory*, **11**(2), (1987), 235-249.

[2] Alison Northup, *A Study of semi regular Graphs*, Preprint, (2002).

[3] G.S.Bloom, J.K.Kennedy and L.V.Quintas, Distance degree regular graphs, *The Theory and Applications of Graphs*, Wiley, New York, (1981), 95-108.

[4] J.A.Bondy and U.S.R.Murty, *Graph Theory with Application*, MacMillan, London (1979).

[5] Gary Chartrand, Paul Erdos, Ortrud R. Oellerman, How to Define an irregular graph, *College Math. Journal*, **39**(1998).

[6] F.Harary, *Graph Theory*, Addison - Wesley, (1969).

[7] K.R.Parthasarathy, *Basic Graph Theory*, Tata McGraw- Hill Publishing company Limited, New Delhi.

[8] N.R.Santhi Maheswari and C.Sekar, $(r, 2, r(r-1))$-Regular graphs, International Journal of Mathematics and Soft Computing, vol. 2, No.2(2012), 25 -33.

Math.Combin.Book Ser. Vol.4(2012), 52-57

Further Results on Global Connected Domination Number of Graphs

G.Mahadevan[1], A.Selvam Avadayappan[2] and Twinkle Johns[3]

1. Department of Mathematics, Anna University of Technology Tirunelveli, Tirunelveli-627007, India

2. Department of Mathematics, V.H.N.S.N.College, Virdhunagar-626204, India

3. Department of Mathematics, V.P.M.M.Engineering College for Women, Krishnankoil-626190, India

E-mail: gmaha2003@yahoo.co.in, selvamavadayappan@yahoo.co.in, twinkle.johns@gmail.com

Abstract: A subset S of vertices in a graph $G=(V,E)$ is a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $< S >$ is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. In this paper we obtained the upper bound for the sum of global connected domination number and chromatic number and characterize the corresponding extremal graphs.

Key Words: Global domination number, chromatic number.

AMS(2010): 05C69

§1. Introduction

Graphs discussed in this paper are simple, finite and undirected graphs. A subset S of vertices in a graph $G=(V,E)$ is a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set S of a connected graph G is called a connected dominating set if the induced subgraph $\langle S\rangle$ is connected. A set S is called a global dominating set of G if S is a dominating set of both G and $\overline{G}$. A subset S of vertices of a graph G is called a global connected dominating set if S is both a global dominating and a connected dominating set. The global connected domination number is the minimum cardinality of a global connected dominating set of G and is denoted by $\gamma_{gc}(G)$. Note that any global connected dominating set of a graph G has to be connected in G (but not necessarily in $\overline{G}$). Here global connected domination number γ_{gc} is well defined for any connected graph. For a cycle C_n of order $n\geq 6$, $\gamma_g(C_n)=\lceil n/3\rceil$ while $\gamma_{gc}(C_n)=n-2$ and $\gamma_g(K_n)=1$, while $\gamma_{gc}(K_n)=n$. The chromatic number $\chi(G)$ is defined as the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color.

[1]Received May 28, 2011. Accepted December 8, 2012.

Notation 1.1 $K_n(P_k)$ is the graph obtained from K_n by attaching the end vertex of P_k to any one vertices of K_n. $K_n(mP_k)$ is the graph obtained from K_n by attaching the end vertices of m copies of P_k to any one vertices of K_n.

Some preliminary results on global connected domination number of graphs are listed in the following.

Theorem 1.2([1]) *Let G be a graph of order $n \geq 2$. Then*

(1) $2 \leq \gamma_{gc}(G) \leq n$,

(2) $\gamma_{gc}(G) = n$ if and only if $G \cong K_n$.

Corollary 1.3([1]) *For all positive integers p and q $\gamma_{gc}(K_{p,q}) = 2$.*

Theorem 1.4([1], Brooke's Theorem) *If G is a connected simple graph and is neither a complete graph nor an odd cycle then $\chi(G) \leq \Delta(G)$.*

Theorem 1.5([1]) *For any graph G of order $n \geq 3$, $\gamma_{gc}(G) = n-1$ if and only if $G \cong K_n - e$, where e is an edge of K_n.*

Corollary 1.6([1]) *For all $n \geq 4$ $\gamma_{gc}(C_n) = n-2$.*

§2. Main Result

Theorem 2.1 *For any connected graph G, $\gamma_{gc}(G) + \chi(G) < 2n-1$.*

Proof By Theorem 1.2, for any graph G of order $n \geq 2$, $\gamma_{gc}(G) \leq n$.

By Theorem 1.4, $\chi(G) \leq \Delta(G)$. Therefore, $\gamma_{gc}(G) + \chi(G) \leq n + \Delta = n + (n-1) = 2n-1$. Hence $\gamma_{gc}(G) + \chi(G) \leq 2n-1$. Let $\gamma_{gc}(G) + \chi(G) = 2n-1$. It is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n-1$ (or) $\gamma_{gc}(G) = n-1$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, G contains a clique K on $n-1$ vertices or does not contain a clique K on $n-1$ vertices. Suppose G contains a clique K on $n-1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected, the vertex v_1 is adjacent to some vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, therefore $K \cong K_1$ which is a contradiction. Hence no graph exist. If G does not contain the clique K on $n-1$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n$. Since $\gamma_{gc}(G) = n-1$ by Theorem 1.5, $G \cong K_n - e$ But for $K_n - e$, $\chi(G) = n-1$ which is a contradiction. Hence no graph exists. Hence, $\gamma_{gc}(G) + \chi(G) < 2n-1$. □

Theorem 2.2 *For any connected graph G, for $n \geq 3$ $\gamma_{gc}(G) + \chi(G) = 2n-2$ if and only if $G \cong K_n - e$, where e is an any edge of K_n.*

Proof Assume that $\gamma_{gc}(G) + \chi(G) = 2n - 2$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 2$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and$\chi(G) = n - 2$. Since $\chi(G) = n - 2$, G contains a clique K on $n - 2$ vertices or does not contain a clique K on $n - 2$ vertices . Suppose G contains a clique K on $n - 2$ vertices. Let $S = \{v_1, v_2\} \subset V - K$. Then the induced sub graph $< S >$ has the following possible cases, $< S >= K_2$ and $\overline{K_2}$.

Subcase 1.1 Let $\langle S \rangle = K_2$. Since G is connected there exist a vertex u_i of K_{n-2} which is adjacent to anyone of v_1, v_2. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ forms a global connected dominating set in G. So that $\gamma_{gc}(G) = 2$ which is a contradiction.

Subcase 1.2 Let $\langle S \rangle = \overline{K_2}$. Since G is connected, let both the vertices of $\overline{K_2}$ be adjacent to vertex u_i for some i in $K_n - 2$. Let $\{v_1, v_2\}$ be the vertices of $\overline{K_2}$. Then anyone of the vertices of $\overline{K_2}$ and u_i forms a global connected dominating set in G. Then, $\{v_1, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. If both the vertices of $\overline{K_2}$ are adjacent two distinct vertices of K_{n-2} say u_i and u_j for $i \neq j$ in K_{n-2}. Then $\{u_i, u_j\}$ forms a global connected dominating set. Hence $\gamma_{gc}(G) = 2$, Which is a contradiction. If G does not contain the clique K on $n-2$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 1$. Since $\gamma_{gc}(G) = n - 1$, by Theorem 1.5, $G \cong K_n - e$. But for $K_n - e$, $\chi(G) = n - 1$. Hence $G \cong K_n - e$ for $n \geq 3$. If G does not contain the clique K on $n - 1$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n$. Since $\chi(G) = n$. Then $G \cong K_n$. But for K_n, $\gamma_{gc}(G) = n$, which is a contradiction.

Conversely if G is anyone of the graph $K_n - e$, for $n \geq 3$ then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n - 2$. □

Theorem 2.3 *For any connected graph G, $\gamma_{gc}(G) + \chi(G) = 2n - 3$ if and only if $G \cong K_3(P_2), K_n - \{e_1, e_2\}$, where e_1, e_2 are edges in the cycle of graph, $n \geq 5$.*

Proof Assume that $\gamma_{gc}(G) + \chi(G) = 2n - 3$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n - 3$ (or) $\gamma_{gc}(G) = n - 1$ and $\chi(G) = n - 2$ (or) $\gamma_{gc}(G) = n - 2$ and $\chi(G) = n - 1$ (or) $\gamma_{gc}(G) = n - 3$ and $\chi(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n - 3$. Since $\gamma_{gc}(G) = n - 3$, G contains a clique K on $n - 3$ vertices or does not contain a clique K on $n - 3$ vertices. Suppose G contains a clique K on $n - 3$ vertices. Let $S = \{v_1, v_2, v_3\} \subset V - K$. Then the induced sub graph $\langle S \rangle$ has the following possible cases $\langle S \rangle = K_3$, $\overline{K_3}$, $K_2 \cup K_1$, P_3.

Subcase 1.1 Let $\langle S \rangle = K_3$. Since G is connected, these exist a vertex u_i of K_{n-3} which is adjacent to anyone $\{v_1, v_2, v_3\}$. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, u_i\}$ is a global connected dominating set. Hence $\gamma_{gc}(G) = 2$ which is a contradiction.

Subcase 1.2 Let $\langle S \rangle = \overline{K_3}$. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to all the vertices of $\overline{K_3}$. Let v_1, v_2, v_3 be the vertices of $\overline{K_3}$. Then anyone of the vertices of $\overline{K_3}$ and u_i forms a global connected dominating set in G. Without loss of generality $\{v_1, u_i\}$

forms a global connected dominating set in G. Then $\gamma_{gc}(G) = 2$, which is a contradiction. If two vertices of $\overline{K_3}$ are adjacent to u_i and third vertex adjacent to u_j for some $i \neq j$. Then $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Then $\gamma_{gc}(G) = 3$, which is a contradiction. If three vertices of $\overline{K_3}$ are adjacent to three distinct vertices of K_{n-3} say u_i, u_j, u_k for some $i \neq j \neq k$. Then $\{v_1, u_i, u_j, u_k\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 4$ which is a contradiction.

Subcase 1.3 Let $\langle S\rangle = K_2 \cup K_1$. Since G is connected, there exist a vertex u_i of K_{n-3} which is adjacent to anyone of $\{v_1, v_2\}$ and v_3. Then $\{v_1, u_i\}$ is a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$, which is a contradiction. Let there exist a vertex u_i of K_{n-3} be adjacent to anyone of $\{v_1, v_2\}$ and u_j for some $i \neq j$ in K_{n-3} adjacent to v_3. Without loss of generality, let v_1 be adjacent to u_i. Then $\{v_1, u_i, u_j\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ which is a contradiction.

Subcase 1.4 Let $\langle S\rangle = P_3$. Let $\{v_1, v_2, v_3\}$ be the vertices of P_3. Since G is connected there exist a vertex u_i of K_{n-3} which is adjacent to anyone of the pendant vertices v_1 or v_3. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1, v_2, u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G) = 3$ which is a contradiction. On increasing the degree of u_i, which is a contradiction. If G does not contain the clique K on $n-3$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-2$. Since $\gamma_{gc}(G) = n-1$ by Theorem 1.5, $G \cong K_n - e$. But for $K_n - e$, $\chi(G) = n-1$, which is a contradiction. Hence no graph exists. If G does not contain the clique K on $n-3$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-1$. Since $\chi(G) = n-1$, G contains a clique K on $n-1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected, the vertex v is adjacent to vertex u_i of K_{n-1}. Then $\{v_1, u_i\}$ is a global connected dominating set in G. Hence $\gamma_{gc}(G) = 2$. Then $K \cong K_3$. Then $G \cong K_3(P_2)$. On increasing the degree of v $G \cong K_n - \{e_1, e_2\}$ for $n \geq 5$ and e_1, e_2 is an edge in outside the cycle of graph. If G does not contain the clique K on $n-1$ vertices, then it can be verified that no graph exists.

Case 4. Let $\gamma_{gc}(G) = n-3$ and $\chi(G) = n$. Since $\chi(G) = n$ then $G \cong K_n$. But for K_n $\gamma_{gc}(G) = n$, then it is a contradiction.

Conversely if G is anyone of the graph $K_3(P_2)$, $K_n - \{e_1, e_2\}$ for $n \geq 5$, then it can be verified that $\gamma_{gc}(G) + \chi(G) = 2n-3$. □

Theorem 2.4 *For any connected graph G $\gamma_{gc}(G) + \chi(G) = 2n-4$ if and only if $G \cong P_4, C_4, K_2(2P_2), K_4(P_2), K_n - \{e_1, e_2, e_3\}$ wheree_1, e_2, e_3 are edges in the cycle of graph of order $n \geq 6$*

Proof Let $\gamma_{gc}(G) + \chi(G) = 2n-4$. This is possible only if $\gamma_{gc}(G) = n$ and $\chi(G) = n-4$ (or) $\gamma_{gc}(G) = n-1$ and $\chi(G) = n-3$ (or) $\gamma_{gc}(G) = n-2$ and $\chi(G) = n-2$ (or) $\gamma_{gc}(G) = n-3$ and $\chi(G) = n-1$ (or) $\gamma_{gc}(G) = n-4$ and $\gamma(G) = n$.

Case 1 Let $\gamma_{gc}(G) = n$ and $\chi(G) = n-4$. Since $\chi(G) = n-4$, G contains a clique K on $n-4$ vertices or does not contain a clique K on $n-4$ vertices. Suppose G contains a clique K on

$n-4$ vertices. Let $S=\{v_1,v_2,v_3,v_4\}\subset V-K$. Then the induced subgraph $<S>$ has the following possible cases.$<S>=K_4,\overline{K_4},K_3\cup K_1,K_4-e,K_2\cup K_2,K_2\cup\overline{K_2},K_3(P_2)$ it can be verified that all the above cases no graph exist.

If G does not contain the clique K on $n-4$ vertices, then it can be verified that no graph exists.

Case 2 Let $\gamma_{gc}(G)=n-1$ and $\chi(G)=n-3$. Since $\gamma_{gc}(G)=n-1$ by Theorem 1.5, $G\cong K_n-e$. But for K_n-e $\chi(G)=n-1$ which is a contradiction.

If G does not contain the clique K on $n-3$ vertices, then it can be verified that no graph exists.

Case 3 Let $\gamma_{gc}(G)=n-2$ and $\chi(G)=n-2$. Since $\chi(G)=n-2$, then G contains a clique K on $n-2$ vertices or does not contain a clique K on $n-2$ vertices. Suppose G contains a clique K on $n-2$ vertices. Let $S=\{v_1,v_2\}\subset V-K$. Then the induced subgraph $<S>$ has the following possible cases,$<S>=K_2$ and $\overline{K_2}$.

Subcase 3.1 Let $\langle S\rangle=K_2$. Since G is connected there exist a vertex u_i of K_{n-2} which is adjacent to anyone of $\{v_1,v_2\}$. Without loss of generality let v_1 be adjacent to u_i. Then $\{v_1,u_i\}$ is a global connected dominating set in G hence $\gamma_{gc}(G)=2$. Then $K\cong K_2$. Let $\{v_1,v_2\}$ be the vertices of K_2. If $d(v_1)=1$, $d(v_2)=2$ Then $G\cong P_4$. If $d(v_1)=2$ $d(v_2)=2$ then $G\cong C_4$.

Subcase 3.2 Let $\langle S\rangle=\overline{K_2}$. Since G is connected. Let both the vertices of $\overline{K_2}$ be adjacent to vertex u_i for some i in K_{n-2}. Let $\{v_1,v_2\}$ be the vertices of $\overline{K_2}$. Then anyone of the vertices of $\overline{K_2}$ and u_i forms a global connected dominating set in G. Without loss of generality v_1 and u_i forms a global connected dominating set in G. Hence $\gamma_{gc}(G)=2$ so that $K\cong K_2$. Then $G\cong K_2(2P_2)$. If two vertices of $\overline{K_2}$ are adjacent two distinct vertices K_{n-2} say u_i and u_j for $i\neq j$ .Then $\{v_1,u_i,u_j\}$ forms global connected dominating set in G. Hence $\gamma_{gc}(G)=3$ so that $K\cong K_2$. Then $G\cong K_3(P_2,P_2,0)$.

If G does not contain the clique K on $n-2$ vertices, then it can be verified that no graph exists.

Case 4 Let $\gamma_{gc}(G)=n-3$ and $\chi(G)=n-1$. Since $\chi(G)=n-1$, G contains a clique K on $n-1$ vertices. Let v be the vertex not in K_{n-1}. Since G is connected the vertex v is adjacent to vertex u_i of K_{n-1}. Then $\{v_1,u_i\}$ forms a global connected dominating set in G. Hence $\gamma_{gc}(G)=2$ so that $K\cong K_4$. Then $G\cong K_4(P_2)$. On increasing the degree $G\cong K_n-\{e_1,e_2,e_3\}$ where e_1,e_2,e_3 are edges in the cycle of the graph of order $n\geq 6$. If G does not contain the clique K on $n-1$ vertices, then it can be verified that no graph exists.

Case 5 Let $\gamma_{gc}(G)=n-4$ and $\chi(G)=n$. Since $\chi(G)=n$. Then $G\cong K_n$. But for K_n, $\gamma_{gc}(Kn)=n$, Which is a contradiction.

Conversely if G is anyone of the graph $P_4,C_4,K_2(2P_2),K_4(P_2),K_n-3e$ for $n\geq 6$ then it can be verified that $\gamma_{gc}(G)+\chi(G)=2n-4$. □

The authors obtained graphs for which $\gamma_{gc}(G)+\chi(G)=2n-5,2n-6,2n-7$, which will be reported later.

References

[1] Dejan Delic and Changping Wang (print), The Global connected domination in Graphs.

[2] E.Sampathkumar, Global domination number of a graph, *J. Math. Phy. Sci.*, 23 (1989) 377-385.

[3] John Clark and Derek Allan Holton (1995), *A First Look at Graph Theory*, Allied Publishers Ltd.

[4] F. Harary (1972), *Graph Theory*, Addison Wesley Reading Mass.

[5] Paulraj Joseph J. and Arumugam S. (1992), Domination and connectivity in graphs, *International Journal of Management and Systems*, Vol.8, No.3, 233-236.

[6] Paulraj Joseph J. and Arumugam S. (1997), Domination and colouring in graphs, *International Journal of Management and Systems*, Vol.8, No.1, 37-44.

[7] Paulraj Joseph J. and Mahadevan G. (2002), Complementary connected domination number and chromatic number of a graph, *Mathematical and Computational Models*, Allied Publications, India 342-349.

[8] Mahadevan G. (2005), *On Domination Theory and Related Topics in Graphs*, Ph.D, Thesis, Manomaniam Sundaranar University, Tirunelveli.

[9] Paulraj Joseph J. and Mahadevan G and Selvam A (2006), On Complementary Perfect domination number of a graph, *Acta Ciencia Indica*, Vol XXXI M, No.2, 847,*An International Journal of Physical Sicences*.

[10] Mahadevan G, Selvam A, Hajmeeral.M (2009), On efficient domination number and chromatic number of a graph I, *International Journal of Physical Sciences*, Vol 21(1)M, pp1-8.

Math.Combin.Book Ser. Vol.4(2012), 58-67

Average Lower Domination Number for Some Middle Graphs

Derya Dogan

(Department of Mathematics, Faculty of Science, Ege University, 35100, Bornova-IZMIR, TURKEY)

E-mail: derya.dogan@ege.edu.tr

Abstract: Communication network is modeled as a simple, undirected, connected and unweighted graph G. Many graph theoretical parameters can be used to describe the stability and reliability of communication networks. If we consider a graph as modeling a network, the average domination number of a graph is one of the parameters for graph vulnerability. In this paper, we consider the average domination number for middle graphs of some well-known graphs, in particular we consider the middle graphs because these graphs are between total graphs and line graphs. In real life problems, every edge corresponds cost, so middle graphs make sense on this situation.

Key Words: Domination, Average lower domination number, middle graphs.

AMS(2010): 05C69

§1. Introduction

The *line graph* $L(G)$ of a graph G is the graph whose vertex set is the edge set $\mathrm{E}(G)$ of G, with two vertices of $L(G)$ being adjacent if and only if the corresponding edges in G have a vertex in common. The *middle graph* $M(G)$ is the graph obtained from G inserting a new vertex into every edge of G and by joining edges those pairs of these new vertices which lie on adjacent edges of G. Another important graph is the total graph. The *total graph* $T(G)$ is the graph whose vertex set is the union of the vertex set $V(G)$ and the edge set $E(G)$ of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident. There have been lots of research on various properties of line graphs, middle graphs and total graphs of graphs.

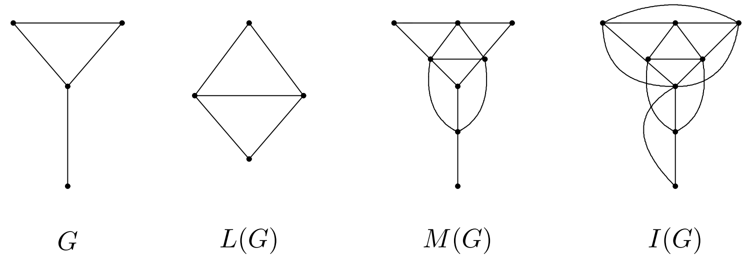


Fig.1

[1]Received July 10, 2012. Accepted December 10, 2012.

The stability of communication network is composed of processing nodes and communication links, is the prime importance of network designers. Graph theoretical parameters can be used to describe the stability and reliability of communication networks. If we consider a graph as modeling a network, the domination number of a graph is one of the parameters for graph vulnerability.

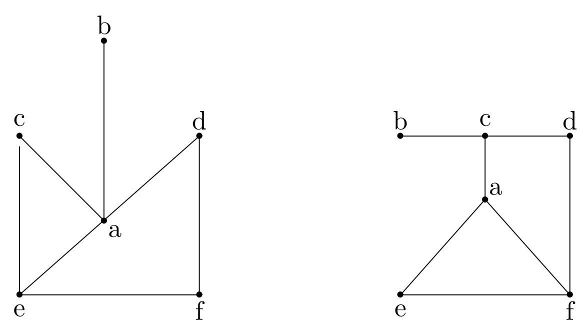
In this paper, we considered average domination number of a graph instead of domination number of a graph. Let's see what difference between these two parameters is and why we consider average domination number instead of domination number.

A vertex v in a graph G said to dominate itself and each of its neighbors, that is, v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices of G is a dominating set of G if every vertex of G is dominated by at least one vertex of S. Equivalently, a set S of vertices of G is dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a minimum dominating set.

Let $G = (V, E)$ be a graph, the domination number $\gamma_v(G)$ of G relative to v is the minimum cardinality of a dominating set in G that contains v. The average domination number of G, $\gamma_{av}(G)$, can be written as:

$$\frac{1}{|V(G)|} \sum_{v \epsilon V(G)} \gamma_v(G).$$

Let H and G be a graphs which are have same order and size as below:



Graph G Graph H

Fig.2

It can easily seen that $\kappa(G) = \kappa(H) = 1$, $\beta(G) = \beta(H) = 3$, $\alpha(G) = \alpha(H) = 3$ and also $\gamma(G) = \gamma(H) = 2$. In this case, *how can we decide which graph is more reliable and how to find the average domination number of these graphs?*

v		γ_v
a	$\{a, f\}$	2
b	$\{b, e, d\}$	3
c	$\{c, a, f\}$	3
d	$\{d, a\}$	2
e	$\{e, a\}$	2
f	$\{f, a\}$	2

$\gamma_{av}(G) = 14/6 = 2.33$

v		γ_v
a	$\{a, c\}$	2
b	$\{b, f\}$	2
c	$\{c, a,\}$	2
d	$\{d, a, b\}$	3
e	$\{e, c\}$	2
f	$\{f, b\}$	2

$\gamma_{av}(H) = 13/6 = 2.16$

So, $\gamma_{av}(H) < \gamma_{av}(G)$. Then we can say graph H is more reliable than graph G. If we consider the graphs $G-\{a\}$ and $H-\{c\}$, then we can see each graph has one isolated vertex and $G-\{a\}$ has P_4 but $H-\{c\}$ contains cycle in it. This means $H-\{c\}$ is more reliable than $G-\{a\}$.

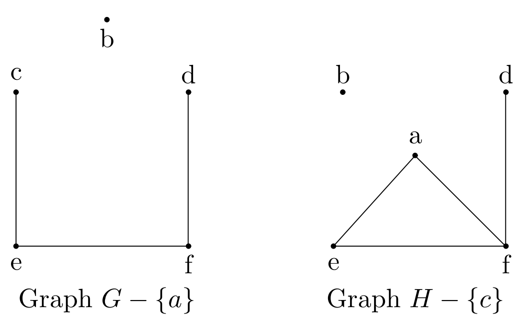


Fig.3

Henning introduced average domination and independent domination numbers, studied trees for which these two parameters are equal. Our goal is to study the average domination number for the middle graphs of some well-known graphs.

§2. Average Domination Numbers for Middle Graphs of Some Well-Known Graphs

Theorem 2.1 *Let $M(K_{1,n})$ be the middle graph of $K_{1,n}$. Then*

$$\gamma_{av}(M(K_{1,n})) = \frac{2n^2+n+1}{2n+1}.$$

Proof The number of vertices of the graph $K_{1,n}$ and $M(K_{1,n})$ are $n+1$ and $2n+1$, respectively. Let say $M(K_{1,n}) = G$. We consider the vertex-set of graph $G = V_1(G) \cup V_2(G) \cup V_3(G)$ where,

$V_1(G)$: The set contains center vertex which has degree n.

$V_2(G)$: The set contains n vertices whose degree is 1.

$V_3(G)$: The set contains n new vertices with degree of $(n+1)$ which are obtained by definition of middle graph.

Then it is easily calculated that the average domination number of graph G. We consider three cases.

Case 1 Let v be the vertex of the $V_1(G)$. The vertex v is the center vertex which have the degree n in $K_{1,n}$. v dominates n vertices which are new vertices of $M(K_{1,n})$. In order to dominate vertices of $V_2(G)$, we have to put these vertices to dominating set including vertex v. Consequently, for the center vertex v, $\gamma_v(G) = n + 1$.

Case 2 Let $v \in V_2(G)$, then degree of v is 1. Since the graph G have n vertices as v, then the cardinality of dominating set including vertex v is n, i.e., $\gamma_v(G) = n$.

Case 3 Let $v \in V_3(G)$. It is similar to that of Case 2. Thus $\gamma_v(G) = n$.

Consequently, we get that

$$\sum_{v\epsilon V(G)} \gamma_v(G) = \sum_{v\epsilon V_1(G)} \gamma_v(G) + \sum_{v\epsilon V_2(G)} \gamma_v(G) + \sum_{v\epsilon V_3(G)} \gamma_v(G)$$

$$\begin{aligned}\gamma_{av}(G) &= \frac{1}{|V(G)|}\Big(\sum_{v\epsilon V_1(G)} \gamma_v(G) + \sum_{v\epsilon V_2(G)} \gamma_v(G) + \sum_{v\epsilon V_3(G)} \gamma_v(G)\Big) \\ &= \frac{1}{2n+1}\left((n+1) + n^2 + n^2\right) = \frac{2n^2+n+1}{2n+1}.\end{aligned}$$ □

Observation 1 *The domination number of $M(P_n)$ is $\left\lceil \frac{n}{2} \right\rceil$.*

Proof We have to take $\left\lceil \frac{n}{2} \right\rceil$ vertices which have degree 4 in our dominating set to dominate n vertices each of which have degree 2. It is easy to see that $\gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$. □

Observation 2 *If $n = 2k$, then the dominating set is unique.*

Proof The proof is clear. □

Theorem 2.2 *Let $M(P_n)$ be the middle graph of P_n. Then*

$$\gamma_{av}(M(P_n)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + \frac{\left\lfloor \frac{2n-1}{4} \right\rfloor}{2n-1}, & n \text{ is odd;} \\ \\ \frac{n^2+n-1}{2n-1}, & n \text{ is even.} \end{cases}$$

Proof Notice that $|V(P_n)| = n$ and $|V(M(P_n))| = 2n - 1$. Let $M(P_n) = G$ and $v_1, v_2, v_3, \cdots, v_{2n-1}$ be the vertices of G. We need to consider two cases which are even and odd order of P_n.

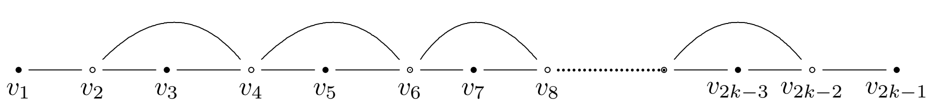


Fig.4

Case 1 If n is odd, then by Observation 1 $\gamma(M(P_n)) = \left\lceil \frac{n}{2} \right\rceil$ and we can find several dominating set that gives us domination number of these vertices (The set is not unique). For the rest of vertices. The cardinality of domination sets which are including v_i such that $i \equiv 3(mod\, 4)$ is $\left\lceil \frac{n}{2} \right\rceil + 1$ and the number of these vertices is $\left\lfloor \frac{2n-1}{4} \right\rfloor$ in $M(P_n)$, where $k \geq 1$. For the rest of vertices, domination number is $\left\lceil \frac{n}{2} \right\rceil$. Then,

$$\begin{aligned}\gamma_{av}(M(P_n)) &= \frac{\left((2n-1) - \left\lfloor \frac{2n-1}{4} \right\rfloor\right)\left(\left\lceil \frac{n}{2} \right\rceil\right) + \left(\left\lfloor \frac{2n-1}{4} \right\rfloor\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right)}{2n-1} \\ &= \left\lceil \frac{n}{2} \right\rceil + \frac{\left\lfloor \frac{2n-1}{4} \right\rfloor}{2n-1}.\end{aligned}$$

Case 2 If n is even, then by Observation 2 dominating set is unique and $\gamma(M(P_n)) = \frac{n}{2}$. It is clear that for the number of $\frac{n}{2}$ vertices we have the same average domination number. For the rest of vertices the dominating number increases 1 unit, so $\gamma_v = \frac{n}{2} + 1$ for the number of $\frac{3n-2}{2}$ vertices. Then,

$$\gamma_{av}(M(P_n)) = \frac{\frac{n}{2} \times \frac{n}{2} + \frac{3n-2}{2}\left(\frac{n}{2} + 1\right)}{2n-1} = \frac{n^2+n-1}{2n-1}.$$

So,

$$\gamma_{av}(M(P_n)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + \frac{\left\lfloor \frac{2n-1}{4} \right\rfloor}{2n-1}, & n \text{ is odd}, \\ \frac{n^2+n-1}{2n-1}, & n \text{ is even}. \end{cases} \qquad \Box$$

Theorem 2.3 *Let $M(C_n)$ be the middle graph of C_n. Then,*

$$\gamma_{av}(M(C_n)) = \frac{n+1}{2}.$$

Proof Notice that $|V(C_n)| = n$, $|V(M(C_n))| = 2n$. Let $M(C_n) = G$. We consider two cases dependent on the parity of $|G|$ following.

Case 1 n is odd.

In this case, we consider two cases, i.e.,$v \in V(C_n)$ and $v \in (V(M(C_n)) \setminus V(C_n)$. First, let $v \in V(C_n)$. Then we want to find γ_v, v dominates 2 new vertices in $M(C_n)$ and itself. The remain vertices which are not dominated gives us P_{n-1}, the domination number of P_{n-1} is $\gamma(M(P_{n-1})) = \frac{n-1}{2}$ by Observation 2. So, $\gamma_v = (1 + \frac{n-1}{2})$ and also we have n vertices as v. If $v \in (V(M(C_n)) \setminus V(C_n)$. Then we want to find γ_v, v dominates 2 new vertices in $M(C_n)$, 2 vertices in C_n and itself. The remain vertices which are not dominated gives us P_{n-2}, the domination number of P_{n-2} is $\gamma(M(P_{n-2})) = \left\lceil \frac{n-2}{2} \right\rceil$ by Observation 1. So, $\gamma_v = (1 + \left\lceil \frac{n-2}{2} \right\rceil)$ and also we have n vertices as v.

By these two subcases,

$$\begin{aligned}\gamma_{av}(M(C_n)) &= \frac{\left(1+(\frac{n-1}{2})\right)n+\left(\left\lceil\frac{n-2}{2}\right\rceil+1\right)n}{2n}\\ &= \left\lceil\frac{n}{2}\right\rceil=\frac{n+1}{2},\end{aligned}$$

where n is odd.

Case 2 n is even.

In this case, we also consider two cases,i.e., $v \in V(C_n)$ and $v \in (V(M(C_n)) \setminus V(C_n)$. First, let $v \in V(C_n)$. Then we want to find γ_v, v dominates 2 new vertices in $M(C_n)$ and itself. The remain vertices which are not dominated gives us P_{n-1}, the domination number of P_{n-1} is $\gamma(M(P_{n-1})) = \left\lceil\frac{n-1}{2}\right\rceil$ by Observation 1. So, $\gamma_v = (1+\left\lceil\frac{n-1}{2}\right\rceil)$ and also we have n vertices as v.

Now if $v \in (V(M(C_n)) \setminus V(C_n)$. Then we want to find γ_v, v dominates 2 new vertices in $M(C_n)$, 2 vertices in C_n and itself. The remain vertices which are not dominated gives us P_{n-2}, the domination number of P_{n-2} is $\gamma(M(P_{n-2})) = \frac{n-2}{2}$ by Observation 1. So, $\gamma_v = (1+\frac{n-2}{2})$ and also we have n vertices as v.

By these two subcases,

$$\gamma_{av}(M(C_n)) = \frac{(1+\left\lceil\frac{n-1}{2}\right\rceil)n+(1+\frac{n-2}{2})n}{2n}=\frac{n+1}{2},$$

where n is even. So,

$$\gamma_{av}(M(C_n)) = \frac{n+1}{2}. \qquad \square$$

Theorem 2.4 *Let $M(W_{1,n})$ be the middle graph of $W_{1,n}$. Then,*

$$\gamma_{av}(M(W_{1,n})) = \begin{cases} \dfrac{\left\lceil\frac{n}{2}\right\rceil(3n+3)}{3n+1}, & n \text{ is odd},\\[2ex] \dfrac{n+2}{2}, & n \text{ is even}.\end{cases}$$

Proof The numbers of vertices of the graph $W_{1,n}$ and $M(W_{1,n})$ are $n+1$ and $3n+1$, respectively. Let $M(W_{1,n}) = G$. We consider the vertex-set of graph $G = V_1(G) \cup V_2(G) \cup V_3(G) \cup V_4(G)$ where,

$V_1(G)$: The set contains center vertex with degree of n of the graph G.

$V_2(G)$: The set contains n vertices whose degree is 3.

$V_3(G)$: The set contains n vertices whose degree is 6.

$V_4(G)$: The set contains n new vertices with degree of $n+3$ which are obtained by definition of middle graph.

Then it is easily calculated that the average domination number of graph G. Now, we have cases and also subcases.

Case 1 Let n be odd.

(i) Let $v \in V_1(G)$. The vertex v dominates vertices of $V_2(G)$. In order to dominate rest of vertices we have to put $\left\lceil \frac{n}{2} \right\rceil$ vertices in our dominating set which is $\gamma(M(C_n))$ by the proof of Case 1 for C_n. Consequently, $\gamma_v(G) = \left\lceil \frac{n+2}{2} \right\rceil$.

(ii) Let $v \in V_2(G)$. The vertex v dominates 3 vertices which are new vertices in $M(W_{1,n})$ and itself. In order to dominate K_{n+1} that is obtain from vertex of $V_1(G)$ and new vertices of $M(W_{1,n})$ which are obtain from incident edges of $V_1(G)$ in $W_{1,n}$, we need one vertex. Then to dominate the rest of vertices we have to put $\left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil - 1$ vertices in our dominating set which is the $\gamma(M(P_{n-2}))$ by Observation 1. Hence,

$$\gamma_v(G) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

(iii) Let $v \in V_3(G)$. The vertex v dominates 4 vertices which are new vertices in $M(W_{1,n})$, 2 of them outside and the other 2 inside of wheel, 2 vertices of $W_{1,n}$ and itself. In order to dominate K_{n+1}, we need one vertex which is belong to K_{n+1}, say u. u dominates 3 other vertices which are not element of $V(K_{n+1})$. So, for the rest of vertices we have to put $\frac{n-3}{2}$ vertices in our dominating set which is the $\gamma(M(P_{n-3}))$ by Observation 2. Hence,$\gamma_v(G) = 1 + 1 + \frac{n-3}{2} = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.

(iv) Let $v \in V_4(G)$. The vertex v dominates K_{n+1}, 3 additional vertices(which are in outside of wheel) and itself. For other vertices, we have to put $\frac{n-1}{2}$ vertices in our dominating set which is the $\gamma(M(P_{n-1}))$ by Observation 2. Hence, $\gamma_v(G) = \frac{n-1}{2} + 1 = \frac{n+1}{2} = \left\lceil \frac{n}{2} \right\rceil$.

By these four subcases, we know that

$$\begin{aligned}
\sum_{v \epsilon V(G)} \gamma_v(G) &= \sum_{v \epsilon V_1(G)} \gamma_v(G) + \sum_{v \epsilon V_2(G)} \gamma_v(G) + \sum_{v \epsilon V_3(G)} \gamma_v(G) + \sum_{v \epsilon V_4(G)} \gamma_v(G) \\
\gamma_{av}(G) &= \frac{1}{|V(G)|}\Big(\sum_{v \epsilon V_1(G)} \gamma_v(G) + \sum_{v \epsilon V_2(G)} \gamma_v(G) + \sum_{v \epsilon V_3(G)} \gamma_v(G) + \sum_{v \epsilon V_4(G)} \gamma_v(G)\Big) \\
&= \frac{\left\lceil \frac{n+2}{2} \right\rceil + \left(\left\lceil \frac{n}{2} \right\rceil + 1\right) n + \left\lceil \frac{n}{2} \right\rceil n + \left\lceil \frac{n}{2} \right\rceil n}{3n+1} = \frac{(3n+3)\left\lceil \frac{n}{2} \right\rceil}{3n+1}.
\end{aligned}$$

Case 2 Let n be even.

(i) Let $v \in V_1(G)$. The vertex v dominates vertices of $V_2(G)$. In order to dominate rest of vertices we have to put $\frac{n}{2}$ vertices in our dominating set which is the $\gamma(M(C_n))$ by the proof of Case 2 for C_n. Consequently, $\gamma_v(G) = \frac{n}{2} + 1$.

(ii) Let $v \in V_2(G)$. The vertex v dominates 3 vertices which are new vertices in $M(W_{1,n})$ and itself. In order to dominate K_{n+1}, we need to put one vertex in our dominating set which is belong to K_{n+1} and also this vertex dominates to more outside vertices. Then to dominate the

rest of vertices we have to put $\frac{n-2}{2}$ vertices in our dominating set which is the $\gamma(M(P_{n-2}))$ by Observation 2. Hence,$\gamma_v(G) = \frac{n+2}{2}$.

(iii) Let $v \in V_3(G)$. The vertex v dominates 4 vertices which are new vertices in $M(W_{1,n})$, 2 of them outside and the other 2 inside of wheel, 2 vertices of $W_{1,n}$ and itself. In order to dominate K_{n+1}, we need one vertex which is belong to K_{n+1}, say u. u dominates 3 other vertices which are not element of $V(K_{n+1})$. So, for the rest of vertices we have to put $\left\lceil \frac{n-3}{2} \right\rceil$ vertices in our dominating set which is the $\gamma(M(P_{n-3}))$ by Observation 1. Hence,

$$\gamma_v(G) = \left\lceil \frac{n-3}{2} \right\rceil + 1 + 1 = \frac{n}{2} + 1.$$

(iv) Let $v \in V_4(G)$. The vertex v dominates K_{n+1}, 3 additional vertices(which are in outside of wheel) and itself. So, for the rest of vertices we have to put $\left\lceil \frac{n-1}{2} \right\rceil$ vertices in our dominating set which is the $\gamma(M(P_{n-1}))$ by Observation 1. Hence,$\gamma_v(G) = \left\lceil \frac{n-1}{2} \right\rceil + 1 = \frac{n}{2} + 1$.

By these four subcases;

$$\begin{aligned}
\sum_{v\epsilon V(G)} \gamma_v(G) &= \sum_{v\epsilon V_1(G)} \gamma_v(G) + \sum_{v\epsilon V_2(G)} \gamma_v(G) + \sum_{v\epsilon V_3(G)} \gamma_v(G) + \sum_{v\epsilon V_4(G)} \gamma_v(G) \\
\gamma_{av}(G) &= \frac{1}{|V(G)|}\Big(\sum_{v\epsilon V_1(G)} \gamma_v(G) + \sum_{v\epsilon V_2(G)} \gamma_v(G) + \sum_{v\epsilon V_3(G)} \gamma_v(G) + \sum_{v\epsilon V_4(G)} \gamma_v(G)\Big) \\
&= \frac{\left(\frac{n}{2}+1\right) + \left(\frac{n+2}{2}\right) n + \left(\frac{n}{2}+1\right) n + \left(\frac{n}{2}+1\right) n}{3n+1} = \frac{n}{2} + 1.
\end{aligned}$$

□

Theorem 2.5 *Let $M(K_n)$ be the middle graph of K_n. Then*

$$\gamma_{av}(M(K_n)) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \text{ is odd}; \\ \frac{n^2+n+4}{2n+2}, & n \text{ is even}. \end{cases}$$

Proof The number of vertices of the graph K_n and $M(K_n)$ are n and $\frac{n^2+n}{2}$, respectively. Let $M(K_n) = G$. We consider the vertex-set of graph $G = V_1(G) \cup V_2(G)$, where

$V_1(G)$: The set of vertices of K_n,
$V_2(G)$: The set of vertices of G which are not in V_1.

We need to consider for even and odd n and also each case will have two subcases, $v \in V_1(G)$ and $v \in V_2(G)$.

Case 1 n is odd.

(i) Let $v \in V_1(G)$. In order to find $\gamma_v(G)$, we have to put v in domination set, then still we have $n-1$ vertices in $V_1(G)$ that are not dominated, since each vertices of $V_2(G)$

dominate to two vertices of $V_1(G)$, we need to add $\frac{n}{2}$ vertices of $V_2(G)$ to domination set. Then, $\gamma_v(G)=\frac{n}{2}+1=\left\lceil\frac{n}{2}\right\rceil$ for all $v\in V_1(G)$.

(ii) Let $v\in V_2(G)$. In order to find $\gamma_v(G)$, we have to put v in domination set, then we dominated two vertices of $V_1(G)$ and also $2n-4$ vertices of $V_2(G)$ which are adjacent to v. We have to put $\frac{n-2}{2}$ vertices of $V_2(G)$ in domination set to dominate $n-2$ vertices of $V_1(G)$ which are not dominated. Thus, $\gamma_v(G)=\frac{n-2}{2}+1=\left\lceil\frac{n}{2}\right\rceil$ for all $v\in V_2(G)$.

By (i) and (ii), we get that

$$\begin{aligned}\sum_{v\epsilon V(G)}\gamma_v(G) &= \sum_{v\epsilon V_1(G)}\gamma_v(G)+\sum_{v\epsilon V_2(G)}\gamma_v(G)\\ &= \frac{1}{|V(G)|}\left(\sum_{v\epsilon V_1(G)}\gamma_v(G)+\sum_{v\epsilon V_2(G)}\gamma_v(G)\right)\\ &= \frac{1}{\frac{n^2+n}{2}}\left(n\left\lceil\frac{n}{2}\right\rceil+\left(\frac{n^2-n}{2}\right)\left\lceil\frac{n}{2}\right\rceil\right)=\left\lceil\frac{n}{2}\right\rceil.\end{aligned}$$

Case 2 n is even.

(i) Let $v\in V_1(G)$. The proof is similar to that of Case 1. In order to find $\gamma_v(G)$, we have to put v in domination set, then still we have $n-1$ vertices in $V_1(G)$ that are not dominated, since each vertices of $V_2(G)$ dominate to two vertices of $V_1(G)$, we need to add $\frac{n-1}{2}$ vertices of $V_2(G)$ to domination set. Since n is even, then $\left\lceil\frac{n-1}{2}\right\rceil$. Then, $\gamma_v(G)=\frac{n}{2}+1=\left\lceil\frac{n}{2}\right\rceil$ for all $v\in V_1(G)$.

(ii) Let $v\in V_2(G)$. In order to dominate n vertices of $V_1(G)$, we have to take $\frac{n}{2}$ vertices of $V_2(G)$. Thus, $\gamma_v(G)=\frac{n}{2}$ for all $v\in V_2(G)$.

By (i) and (ii),

$$\begin{aligned}\sum_{v\epsilon V(G)}\gamma_v(G) &= \sum_{v\epsilon V_1(G)}\gamma_v(G)+\sum_{v\epsilon V_2(G)}\gamma_v(G)\\ &= \frac{1}{|V(G)|}\left(\sum_{v\epsilon V_1(G)}\gamma_v(G)+\sum_{v\epsilon V_2(G)}\gamma_v(G)\right)\\ &= \frac{1}{\frac{n^2+n}{2}}\left(n\,(\frac{n}{2}+1)+\left(\frac{n^2-n}{2}\right)\frac{n}{2}\right)=\frac{n^2+n+4}{2n+2}.\end{aligned}$$

By Cases 1 and 2, the average domination number of $M(K_n)$ is,

$$\gamma_{av}(M(K_n)) = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil, & \text{n is odd;} \\ \\ \dfrac{n^2+n+4}{2n+2}, & n \text{ is even.} \end{cases} \qquad \square$$

References

[1] Blidia M., Chellali M. and Ma_ray F., On average lower independence and domination numbers in graphs, *Discrete Math.*, 295 (2005), 1-11.

[2] Chartrand G. and Lesniak L., *Graphs & Digraphs* (fourth ed.), Chapman & Hall, Boca Raton, FL, 2005.

[3] Haynes T.W., Hedetniemi S.T. and Slater P.J. (Eds.), *Fundamentals of Domination in Graphs*, Marcel Dekker, New York (1998)

[4] Haynes T.W., Hedetniemi S.T. and Slater P.J. (Eds.), *Domination in Graphs: Advanced Topic,.* Marcel Dekker, New York (1998).

[5] Henning M.A., Trees with equal average domination and independent domination numbers, *Ars Combin.*, 71 (2004) 305{318.

[6] Prather R.E., *Discrete Mathematical Structures for Computer Science*, Houghton Mi_in, Boston, (1976).

[7] Sun L. and Wang J., An upper bound for the independent domination number, *J. Combin., Theory*, Series B76, 240-246, 1999.

[8] Lesniak L. and Chartrand G., *Graphs and Digraphs*, California Wadsworth and Brooks, 1986.

Math.Combin.Book Ser. Vol.4(2012), 68-79

Chebyshev Polynomials and Spanning Tree Formulas

S.N.Daoud[1,2]

1. Department of Mathematics, Faculty of Science, El-Minufiya University, Shebeen El-Kom, Egypt

2. Department of Applied Mathematics, Faculty of Applied Science, Taibah University, Al-Madinah, K.S.A.

E-mail: sa_na_daoud@yahoo.com

Abstract: The number of spanning trees in graphs (networks) is an important invariant, it is also an important measure of reliability of a network. In this paper we derive simple formulas of the complexity, number of spanning trees, of some new graphs generated by a new operation, using linear algebra, Chebyshev polynomials and matrix analysis techniques.

Key Words: Number of spanning trees, Chebyshev polynomials, Kirchhoff matrix .

AMS(2010): 33C45

§1. Introduction

In this work we deal with simple and finite undirected graphs $G = (V, E)$, where V is the vertex set and E is the edge set. For a graph G, a spanning tree in G is a tree which has the same vertex set as G. The number of spanning trees in G, also called, the complexity of the graph, denoted by $\tau(G)$, is a well-studied quantity (for long time). A classical result of Kirchhoff in [1] can be used to determine the number of spanning trees for $G = (V, E)$. Let $V = \{v_1, v_2, \cdots, v_n\}$, then the Kirchhoff matrix H defined as an $n \times n$ characteristic matrix $H = D - A$, where D is the diagonal matrix of the degrees of G and A is the adjacency matrix of G, $H = [a_{ij}]$ defined as follows:

(i) $a_{ij} = -1$ if v_i and v_j are adjacent and $i \neq j$;
(ii) a_{ij} equals to the degree of vertex v_i if $i = j$;
(iii) $a_{ij} = 0$, otherwise.

All of co-factors of H are equal to $\tau(G)$. There are other methods for calculating $\tau(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_p$ denote the eignvalues of H matrix of a p point graph. Then it is easily shown that $\mu_p = 0$. Furthermore, Kelmans and Chelnokov [2] shown that $\tau(G) = \dfrac{1}{p}\prod_{k=1}^{p-1}\mu_k$. The formula for the number of spanning trees in a d-regular graph G can be expressed as $\tau(G) = \dfrac{1}{p}\prod_{k=1}^{p-1}(d - \lambda_k)$, where $\lambda_0 = d, \lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ are the eigenvalues of the corresponding adjacency matrix of the graph. However, for a few special families of graphs there exists simple formulas that make it much easier to calculate and determine the number of corresponding

[1]Received November 15, 2011. Accepted December 12, 2012.

spanning trees especially when these numbers are very large. One of the first such result is due to Cayley [3] who showed that complete graph on n vertices, K_n has n^{n-2} spanning trees that he showed $\tau(K_n) = n^{n-2},\ n \geq 2$. Another result, $\tau(K_{p,q}) = p^{q-1}q^{p-1},\ p \geq 1, q \geq 1$, where $K_{p,q}$ is the complete bipartite graph with bipartite sets containing p and q vertices, respectively. It is well known, as in e.g., [4,5]. Another result is due to Sedlacek [6] who derived a formula for the wheel W_{n+1} on $n+1$ vertices, he showed that

$$\tau(W_{n+1}) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$$

for $n \geq 3$. Sedlacek [7] also later derived a formula for the number of spanning trees in a Mobius ladder M_n,

$$\tau(M_n) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right]$$

for $n \geq 2$. Another class of graphs for which an explicit formula has been derived is based on a prism. Boesch, et .al. [8,9].

§2. Chebyshev Polynomial

In this section we introduce some relations concerning Chebyshev polynomials of the first and second kind which we use it in our computations. We begin from their definitions, Yuanping, et. al. [10].

Let $A_n(x)$ be an $n \times n$ matrix such that:

$$A_n(x) = \begin{pmatrix} 2x & -1 & 0 & \cdots & \cdots \\ -1 & 2x & -1 & \ddots & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & \ddots & \ddots & -1 \\ \cdots & \ddots & 0 & -1 & 2x \end{pmatrix},$$

where all other elements are zeros. Further we recall that the Chebyshev polynomials of the first kind are defined by

$$T_n(x) = \cos(n \arccos x). \tag{1}$$

and the Chebyshev polynomials of the second kind are defined by

$$U_{n-1}(x) = \frac{1}{n}\frac{d}{dx}T_n(x) = \frac{\sin(n \arccos x)}{\sin(\arccos x)}. \tag{2}$$

It is easily verified that

$$U_n(x) - 2xU_{n-1}(x) + U_{n-2}(x) = 0 \tag{3}$$

and it can then be shown from this recursion that by expanding $\det A_n(x)$ one gets

$$U_n(x) = \det(A_n(x)), \quad n \geq 1. \tag{4}$$

Furthermore by using standard methods for solving the recursion (3), one obtains the explicit formula:

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[\left(x+\sqrt{x^2-1}\right)^{n+1} - \left(x-\sqrt{x^2-1}\right)^{n+1}\right], \quad n \geq 1, \tag{5}$$

where the identity is true for all complex x (except at $x = \pm 1$ where the function can be taken as the limit). The definition of $U_n(x)$ easily yields its zeros and it can therefore be verified that

$$U_{n-1}(x) = 2^{n-1}\prod_{j=1}^{n-1}\left(x - \cos\frac{j\pi}{n}\right). \tag{6}$$

One further notes that

$$U_{n-1}(-x) = (-1)^{n-1}U_{n-1}(x). \tag{7}$$

These two results yield another formula for $U_n(x)$ following,

$$U_{n-1}^2(x) = 4^{n-1}\prod_{j=1}^{n-1}\left(x^2 - \cos^2\frac{j\pi}{n}\right). \tag{8}$$

Finally, simple manipulation of the above formula yields the following, which also will be extremely useful to us latter:

$$U_{n-1}^2\left(\sqrt{\frac{x+2}{4}}\right) = \prod_{j=1}^{n-1}\left(x - 2\cos\frac{2j\pi}{n}\right). \tag{9}$$

Furthermore, one can shows that

$$U_{n-1}^2(x) = \frac{1}{2(1-x^2)}\left[1 - T_{2n}\right] = \frac{1}{2(1-x^2)}\left[1 - T_n\left(2x^2-1\right)\right]. \tag{10}$$

and

$$T_n(x) = \frac{1}{2}\left[\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right]. \tag{11}$$

§3. Results

In mathematics, one always tries to get new structures from given ones. This also applies to the realm of graphs, where one can generate many new graphs from given set of graphs.

Definition 3.1 *If G_1 and G_2 are vertex-disjoint graphs of the same number of vertices. Then the symmetric join, $G_1 * G_2$ is the graph of $G_1 + G_2$, in which each vertex of G_1 is adjacent to every vertex of G_2 except the paired ones. See Fig.1 for details.*

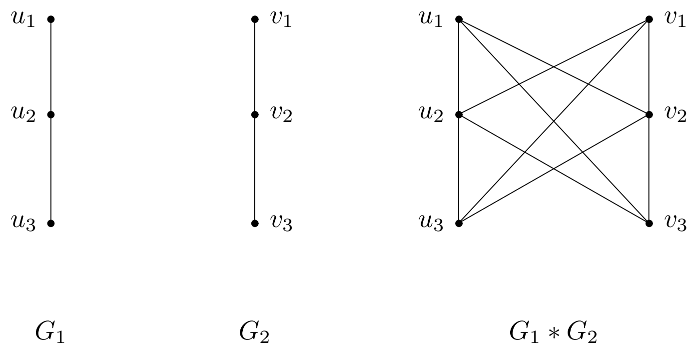


Fig.1

Now, we can introduce the following lemma.

Lemma 3.2 $\tau(G) = \dfrac{1}{p^2}\det(pI - \overline{D} + \overline{A})$, *where* $\overline{A}, \overline{D}$ *are the adjacency and degree matrices of* $\overline{G}$, *the complement of* G, *respectively, and* I *is the* $p \times p$ *unit matrix.*

Proof Straightforward by properties of determinants, matrices and matrix-tree theorem.□

The advantage of these formulas in Lemma 3.2 is to express $\tau(G)$ directly as a determinant rather than in terms of cofactors as in Kirchhoff theorem or eigenvalues as in Kelmans and Chelnokov formula.

Lemma 3.3 *Let* $A \in F^{n\times n}, B \in F^{n\times m}, C \in F^{m\times n}$ *and* $D \in F^{m\times m}$. *If* D *is non-singular, then*

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (-1)^{nm}\det\left(A - BD^{-1}C\right)\det D.$$

Proof It is easy to see that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Thus

$$\begin{aligned} \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \det\begin{pmatrix} A - BD^{-1}C & B \\ 0 & D \end{pmatrix}\det\begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \\ &= (-1^{nm})\det\left(A - BD^{-1}C\right)\det D. \end{aligned}$$ □

This formula gives some sort of symmetry in some matrices which facilitates our calculation of determinants.

Theorem 3.4 *For any integer $n \geq 3$,*

$$\begin{aligned}\tau(P_n * P_n) &= \frac{n-1}{2^{2n} n \sqrt{(n^2-4)\left((n+2)^2-4\right)}} \times \left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right] \\ &\quad \times \left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right].\end{aligned}$$

Proof Applying Lemma 3.2, we get that

$$\tau(P_n * P_n) = \frac{1}{(2n)^2} \det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \left|\begin{array}{cccccccccc}
n+1 & 0 & 1 & \cdots & 1 & & & & & \\
0 & n+2 & 1 & \ddots & \vdots & & & & & \\
1 & 0 & \ddots & \ddots & 1 & & & I & & \\
\vdots & \ddots & \ddots & n+2 & 0 & & & & & \\
1 & \cdots & 1 & 0 & n+1 & & & & & \\
 & & & & & n+1 & 0 & 1 & \ddots & 1 \\
 & & & & & 0 & n+2 & 0 & \ddots & \vdots \\
 & & I & & & 1 & 0 & \ddots & \ddots & \vdots \\
 & & & & & \vdots & \ddots & \ddots & \ddots & 1 \\
 & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\
 & & & & & 1 & \cdots & 1 & 0 & n+1
\end{array}\right|$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix}
n & 0 & 1 & \cdots & 1 \\
0 & n+1 & 0 & \ddots & \vdots \\
1 & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & n+1 & 0 \\
1 & \ddots & 1 & 0 & n
\end{vmatrix} \times \begin{vmatrix}
n+2 & 0 & 1 & \cdots & 1 \\
0 & n+3 & 0 & \ddots & \vdots \\
1 & 0 & \ddots & \ddots & 1 \\
\vdots & \ddots & \ddots & n+3 & 0 \\
1 & \cdots & 1 & 0 & n+2
\end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2}$$

$$\times \begin{vmatrix}
n-1 & -1 & 0 & \cdots & 0 \\
-1 & n & -1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & n & -1 \\
0 & \cdots & 0 & -1 & n-1
\end{vmatrix} \times 2 \begin{vmatrix}
n+1 & -1 & 0 & \cdots & 0 \\
-1 & n+2 & -1 & \ddots & \vdots \\
0 & -1 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & n+2 & -1 \\
0 & \cdots & 0 & -1 & n+1
\end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2} \times (n-2)$$

$$\times \begin{vmatrix} n & -1 & 0 & \cdots & \cdots \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & 0 & -1 & n \end{vmatrix}_{(n-1)\times(n-1)} \times 2n \begin{vmatrix} n+2 & -1 & 0 & \cdots & \cdots \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ \vdots & \cdots & 0 & -1 & n+2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{1}{(2n)^2} \times \frac{2n-2}{n-2}\left[(n-2)U_{n-1}\left(\frac{n}{2}\right)\right] \times \left[2nU_{n-1}\left(\frac{n+2}{2}\right)\right]$$

$$= \frac{1}{(2n)^2} \times \frac{2(n-1)}{n-2} \times (n-2) \times \frac{1}{2^n\sqrt{n^2-4}}\left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right]$$

$$\times \frac{2n}{2^n\sqrt{(n+2)^2+4^2}}\left[\left(n+2+\sqrt{(n-2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right]$$

$$= \frac{n-1}{2^{2n}n\sqrt{(n^2-4)((n+2)^2-4)}} \times \left[\left(n+\sqrt{n^2-4}\right)^n - \left(n-\sqrt{n^2-4}\right)^n\right]$$

$$\times \left[\left(n+2+\sqrt{(n-2)^2-4}\right)^n - \left(n+2-\sqrt{(n+2)^2-4}\right)^n\right].$$

□

Theorem 3.5 *For any integer $n \geq 3$,*

$$\tau(N_n * N_n) = n^{n-2}(n-1)(n-2)^{n-1}.$$

Proof Applying Lemma 3.2, we have

$$\tau(N_n * N_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & \cdots & 1 & & & & \\ 1 & \ddots & \ddots & \vdots & & & I & \\ \vdots & \ddots & \ddots & 1 & & & & \\ 1 & \cdots & 1 & n & & & & \\ & & & & n & 1 & \cdots & 1 \\ & & I & & 1 & \ddots & \ddots & \vdots \\ & & & & \vdots & \ddots & \ddots & 1 \\ & & & & 1 & \cdots & 1 & n \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n+1 & 1 & 1 & \cdots & 1 \\ 1 & n+1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \cdots & \cdots & n+1 & 1 \\ 1 & \cdots & 1 & 1 & n+1 \end{vmatrix} \times \begin{vmatrix} n-1 & 1 & 1 & \cdots & 1 \\ 1 & n-1 & 1 & \ddots & \vdots \\ 1 & 1 & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & n-1 & 1 \\ 1 & \cdots & 1 & 1 & n-1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \prod_{j=1}^{n} ((n+1) + \omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}) \times \prod_{j=1}^{n} ((n-1) + \omega_j + \omega_j^2 + \cdots + \omega_j^{n-1})$$

$$= \frac{1}{(2n)^2} ((n+1) + 1 + \cdots + 1) \times \prod_{j=1, \omega_j \neq 1}^{n} ((n+1) + \underbrace{\omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}}_{=-1})$$

$$\times ((n-1) + 1 + 1 + \cdots + 1) \times \prod_{j=1, \omega_j \neq 1}^{n} ((n-1) + \underbrace{\omega_j + \omega_j^2 + \cdots + \omega_j^{n-1}}_{=-1})$$

$$= \frac{1}{(2n)^2} (n+1+n-1) \times (n+1-1)^{n-1} \times (n-1+n-1) \times (n-1-1)^{n-1}$$

$$= \frac{1}{(2n)^2} \times 2n^n \times 2(n-1)(n-2)^{n-1} = n^{n-2}(n-1)(n-2)^{n-1}. \qquad \square$$

Theorem 3.6 *For any integer $n \geq 3$,*

$$\tau(N_n * P_n) = \frac{n-1}{2^n n \sqrt{n^4 - 8n^2 + 8n}} \times \left[\left(n^2 - 2 + \sqrt{n^4 - 8n^2 + 8n} \right)^n - \left(n^2 - 2 - \sqrt{n^4 - 8n^2 + 8n} \right)^n \right].$$

Proof Applying Lemma 3.2, we get that

$$\tau(N_n * P_n) = \frac{1}{(2n)^2} \det \left(2n \times I - \overline{D} + \overline{A} \right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & 1 & \cdots & 1 & & & & & \\ 1 & n & 1 & \ddots & \vdots & & & & & \\ 1 & 1 & \ddots & \ddots & 1 & & & I & & \\ \vdots & \ddots & \ddots & n & 1 & & & & & \\ 1 & \cdots & 1 & 1 & n & & & & & \\ & & & & & n+1 & 0 & 1 & \cdots & 1 \\ & & & & & 0 & n+2 & 0 & \ddots & \vdots \\ & & I & & & 1 & 0 & \ddots & \ddots & 1 \\ & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\ & & & & & 1 & \cdots & 1 & 0 & n+1 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n^2+2n-3 & 2n-1 & 3n-2 & \cdots & \cdots & 3n-2 \\ 2n-1 & n^2+3n-4 & 2n-1 & 3n-2 & \ddots & \vdots \\ 3n-2 & 2n-1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ \vdots & \ddots & \ddots & \ddots & n^2+3n-4 & 2n-1 \\ 3n-2 & \cdots & \cdots & 3n-2 & 2n-1 & n^2+2n-3 \end{vmatrix}$$

$$= \frac{(n-1)^n}{(2n)^2} \times \begin{vmatrix} n+3 & \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \cdots & \frac{3n-2}{n-1} \\ \frac{2n-1}{n-1} & n+4 & \ddots & \ddots & \ddots & \vdots \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \vdots & \ddots & \ddots & \ddots & n+4 & \frac{2n-1}{n-1} \\ \frac{3n-2}{n-1} & \cdots & \cdots & \frac{3n-2}{n-1} & \frac{2n-1}{n-1} & n+3 \end{vmatrix}.$$

Straight forward induction by using properties of determinants. We obtain that

$$\tau(N_n * P_n) = \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \begin{vmatrix} \frac{n^2-2}{n-1} & 0 & 1 & \cdots & \cdots & 1 \\ 0 & \frac{n^2-2}{n-1}+1 & 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1}+1 & 0 \\ 1 & \cdots & \cdots & 1 & 0 & \frac{n^2-2}{n-1} \end{vmatrix}$$

$$= \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2n-3}{n-2} \begin{vmatrix} \frac{n^2-2}{n-1}-1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & \frac{n^2-2}{n-1} & -1 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1} & -1 \\ 1 & \cdots & \cdots & 0 & -1 & \frac{n^2-2}{n-1}-1 \end{vmatrix}$$

$$= \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2n-3}{n-2} \times \left(\frac{n^2-2}{n-1} - 2\right) \begin{vmatrix} \frac{n^2-2}{n-1} - 1 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & \frac{n^2-2}{n-1} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{(n-1)^{n+1}}{(n-2)n^2} \times \left(\frac{n^2-2}{n-1} - 2\right) \times U_{n-1}\left(\frac{n^2-2}{2(n-1)}\right) = \frac{(n-1)^n}{n} \times U_{n-1}\left(\frac{n^2-2}{2(n-1)}\right)$$

$$= \frac{n-1}{2^n n\sqrt{n^4-8n^2+8n}} \times \left[\left(n^2-2+\sqrt{n^4-8n^2+8n}\right)^n - \left(n^2-2-\sqrt{n^4-8n^2+8n}\right)^n\right]. \square$$

Lemma 3.7 *Let $B_n(x)$ be an $n \times n,\ n \geq 3$ matrix such that*

$$B_n(x) = \begin{pmatrix} x & 0 & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 1 & \cdots & 0 & x \end{pmatrix}$$

with $x \geq 4$. Then

$$\det(B_n(x)) = \frac{2(x+n-3)}{x-3}\left[T_n\left(\frac{x-1}{2} - 1\right)\right].$$

Proof Straightforward induction using properties of determinants and above mentioned definitions of Chebyshev polynomial of the first and second kind. $\square$

Theorem 3.8 *For any integer $n \geq 3$,*

$$\tau(N_n * C_n) = \frac{(n-1)^{n+1}}{2^n n^2(n-2)}$$
$$\times \left[\left(\frac{n^2-2}{n-1} + \sqrt{\left(\frac{n^2-2}{n-1}\right)^2 - 4}\right)^n + \left(\frac{n^2-2}{n-1} - \sqrt{\left(\frac{n^2-2}{n-1}\right)^2 - 4}\right)^n - 2^{n+1}\right].$$

Proof Applying Lemma 3.2, we know that

$$\tau(N_n * C_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n & 1 & 1 & \cdots & 1 & & & & & \\ 1 & n & 1 & \ddots & \vdots & & & & & \\ 1 & 1 & \ddots & \ddots & 1 & & & I & & \\ \vdots & \ddots & \ddots & n & 1 & & & & & \\ 1 & \ddots & 1 & 1 & n & & & & & \\ & & & & & n+2 & 0 & 1 & \cdots & 0 \\ & & & & & 0 & n+2 & 0 & \ddots & \vdots \\ & & & I & & 1 & 0 & \vdots & \ddots & 1 \\ & & & & & \vdots & \ddots & \ddots & n+2 & 0 \\ & & & & & 0 & \cdots & 1 & 0 & n+2 \end{vmatrix}$$

$$= \frac{1}{(2n)^2} \times \begin{vmatrix} n^2+3n-4 & 2n-1 & 3n-2 & \cdots & 3n-2 & 2n-1 \\ 2n-1 & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ 3n-2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 3n-2 \\ 3n-2 & \ddots & \ddots & \ddots & \ddots & 2n-1 \\ 2n-1 & 3n-2 & \cdots & \cdots & 2n-1 & n^2+3n-4 \end{vmatrix}$$

$$= \frac{(n-1)^n}{(2n)^2} \times \begin{vmatrix} n+4 & \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \frac{3n-2}{2n-1} & \frac{2n-1}{n-1} \\ \frac{2n-1}{n-1} & n+4 & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{3n-2}{n-1} \\ \frac{3n-2}{n-1} & \ddots & \ddots & \ddots & n+4 & \frac{2n-1}{n-1} \\ \frac{2n-1}{n-1} & \frac{3n-2}{n-1} & \cdots & \frac{3n-2}{n-1} & \frac{2n-1}{n-1} & n+4 \end{vmatrix}.$$

Straight forward induction by using properties of determinants. We obtain that

$$\tau(N_n * C_n) = \frac{(n-1)^n}{(2n-3)n^2} \times \begin{vmatrix} \frac{n^2-2}{n-1}+1 & 0 & 1 & \cdots & 1 & 0 \\ 0 & \frac{n^2-2}{n-1}+1 & 0 & 1 & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & 1 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & \frac{n^2-2}{n-1}+1 & 0 \\ 0 & 1 & \cdots & 1 & 0 & \frac{n^2-2}{n-1}+1 \end{vmatrix}.$$

Applying Lemma 3.7, we get that

$$\tau(N_n * C_n) = \frac{(n-1)^{n+1}}{(2n-3)n^2} \times \frac{2(n-3)}{n-2}\left[T_n\left(\frac{n^2-2}{2(n-1)}\right)-1\right]$$
$$= \frac{(n-1)^{n+1}}{2^n n^2(n-2)}$$
$$\times\left[\left(\frac{n^2-2}{n-1}+\sqrt{\left(\frac{n^2-2}{n-1}\right)^2-4}\right)^n+\left(\frac{n^2-2}{n-1}-\sqrt{\left(\frac{n^2-2}{n-1}\right)^2-4}\right)^n-2^{n+1}\right]. \qquad \square$$

theorem 3.9 *For any integer $n \geq 3$,*

$$\begin{aligned}\tau(C_n * C_n) &= \frac{n-1}{2^n n^2(n-2)}\left[\left(n+\sqrt{n^2-4}\right)^n+\left(n-\sqrt{n^2-4}\right)^n-2^{n+1}\right]\\ &\quad \times\left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n+\left(n+2-\sqrt{(n+2)^2-4}\right)^n-2^{n+1}\right].\end{aligned}$$

Proof applying Lemma 3.2, we get that

$$\tau(C_n * C_n) = \frac{1}{(2n)^2}\det\left(2n \times I - \overline{D} + \overline{A}\right)$$

$$= \frac{1}{(2n)^2} \times \left|\begin{array}{cccccc|cccccc} n+2 & 0 & 1 & \cdots & 1 & 0 & & & & & & \\ 0 & n+2 & 0 & \ddots & \ddots & 1 & & & & & & \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots & & & I & & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & & & & & & \\ 1 & \ddots & \ddots & \ddots & n+2 & 0 & & & & & & \\ 0 & 1 & \cdots & 1 & 0 & n+2 & & & & & & \\ & & & & & & n+2 & 0 & 1 & \cdots & \cdots & 0 \\ & & & & & & 0 & n+2 & 0 & \ddots & \ddots & 1 \\ & & & & & & 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ & & & & I & & \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ & & & & & & 1 & \ddots & \ddots & \ddots & n+2 & 0 \\ & & & & & & 0 & 1 & \cdots & \cdots & 0 & n+2 \end{array}\right|$$

$$= \frac{1}{(2n)^2} \times \left|\begin{array}{cccccc} n+1 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & n+1 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+1 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & n+1 \end{array}\right| \times \left|\begin{array}{cccccc} n+3 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & n+3 & 0 & \ddots & \ddots & 1 \\ 1 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & \ddots & \ddots & \ddots & n+3 & 0 \\ 0 & 1 & \cdots & \cdots & 0 & n+3 \end{array}\right|.$$

Applying Lemma 3.7, we get that

$$\begin{aligned}\tau(C_n * C_n) &= \frac{1}{(2n)^2} \times \frac{4(n-1)}{n-2}\left[T_n\left(\frac{n}{2}\right)-1\right] \times 4\left[T_n\left(\frac{n+2}{2}\right)-1\right] \\ &= \frac{n-1}{2^n n^2(n-2)}\left[\left(n+\sqrt{n^2-4}\right)^n + \left(n-\sqrt{n^2-4}\right)^n - 2^{n+1}\right] \\ &\quad \times \left[\left(n+2+\sqrt{(n+2)^2-4}\right)^n + \left(n+2-\sqrt{(n+2)^2-4}\right)^n - 2^{n+1}\right]. \quad \square\end{aligned}$$

§4. Conclusion

The number of spanning trees $\tau(G)$ in graphs (networks) is an important invariant. The evaluation of this number is not only interesting from a mathematical (computational) perspective, but also, it is an important measure of reliability of a network and designing electrical circuits. Some computationally hard problems such as the travelling salesman problem can be solved approximately by using spanning trees. Due to the high dependence of the network design and reliability on the graph theory we introduced the above important theorems and their proofs.

References

[1] Kirchhoff G.G., Uber die Auflosung der Gleichungen, auf welche man beider Untersuchung der Linearen Verteilung galvanischer Strme gefhrt wird, *Ann. Phys. Chem.*, 72, 497-508 (1847).

[2] Kelmans A.K. and Chelnokov V.M., Certain polynomials of a graph and graphs with an extremal number of trees, *J. Comb. Theory*(B), 16, 197-214 (1974).

[3] Cayley G.A., A theorem on trees, *Quart. J. Math.*, 23, 276-378 (1889).

[4] Clark L., On the enumeration of multipartite spanning trees of the complete graph, *Bull. Of the ICA*, 38, 50-60 (2003).

[5] Qiao N. S. and Chen B., The number of spanning trees and chains of graphs, *J. Applied Mathematics*, No.9, 10-16 (2007).

[6] Sedlacek J., On the skeleton of a graph or digraph, in *Combinatorial Structures and their Applications* (r. Guy, M. Hanani, N. Saver and J. Schonheim, eds), Gordon and Breach, New York (1970), 387-391.

[7] Sedlacek J., Lucas number in graph theory, *Mathematics (Geometry and Graph Theory)* (Chech), Univ. Karlova, Prague 111-115 (1970).

[8] Boesch F.T. and Bogdanowicz Z.R., The number of spanning trees in a prism, *Inter. J. Comput. Math.*, 21, (1987) 229-243.

[9] Boesch F.T. and Prodinger H., Spanning tree formulas and Chebyshev polynomials, *J. of Graphs and Combinatorics* 2, 191-200 (1986).

[10] Yuanping Z., Xuerong Y. and Mordecai J., Chebyshev polynomials and spanning trees formulas for circulant and related graphs, *Discrete Mathematics*, 298, 334-364 (2005).

Math.Combin.Book Ser. Vol.4(2012), 80-91

Joint-Tree Model and the Maximum Genus of Graphs

Guanghua Dong[1,2], Ning Wang[3], Yuanqiu Huang[1] and Yanpei Liu[4]

1. Department of Mathematics, Normal University of Hunan, Changsha, 410081, China
2. Department of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, China
3. Department of Information Science and Technology, Tianjin University of Finance and Economics, Tianjin, 300222, China
4. Department of Mathematics, Beijing Jiaotong University, Beijing 100044, China

E-mail: gh.dong@163.com, ninglw@163.com, hyqq@hunnu.edu.cn, ypliu@bjtu.edu.cn

Abstract: The vertex v of a graph G is called a 1-*critical-vertex* for the maximum genus of the graph, or for simplicity called 1-*critical-vertex*, if $G-v$ is a connected graph and $\gamma_M(G-v)=\gamma_M(G)-1$. In this paper, through the *joint-tree* model, we obtained some types of 1-*critical-vertex*, and get the upper embeddability of the *Spiral* S_m^n.

Key Words: Joint-tree, maximum genus, graph embedding, Smarandache $\mathscr{P}$-drawing.

AMS(2010): 05C10

§1. Introduction

In 1971, Nordhaus, Stewart and White [12] introduced the idea of the maximum genus of graphs. Since then many researchers have paid attention to this object and obtained many interesting results, such as the results in [2-8,13,15,17] etc. In this paper, by means of the joint-tree model, which is originated from the early works of Liu ([8]) and is formally established in [10] and [11], we offer a method which is different from others to find the maximum genus of some types of graphs.

Surfaces considered here are compact 2-dimensional manifolds without boundary. An orientable surface S can be regarded as a polygon with even number of directed edges such that both a and a^{-1} occurs once on S for each $a \in S$, where the power "-1"means that the direction of a^{-1} is opposite to that of a on the polygon. For convenience, a polygon is represented by a linear sequence of lowercase letters. An elementary result in algebraic topology states that

[1]This work was partially Supported by the China Postdoctoral Science Foundation funded project (Grant No: 20110491248), the New Century Excellent Talents in University (Grant No: NCET-07-0276), and the National Natural Science Foundation of China (Grant No: 11171114).

[2]Received August 13, 2012. Accepted December 15, 2012.

each orientable surface is equivalent to one of the following standard forms of surfaces:

$$O_p = \begin{cases} a_0a_0^{-1}, & p = 0, \\ \prod\limits_{i=1}^{p} a_ib_ia_i^{-1}b_i^{-1}, & p \geq 1\ . \end{cases}$$

which are the sphere ($p = 0$), torus ($p = 1$), and the orientable surfaces of genus p ($p \geq 2$). The genus of a surface S is denoted by $g(S)$. Let A, B, C, D, and E be possibly empty linear sequence of letters. Suppose $A = a_1a_2\ldots a_r, r \geq 1$, then $A^{-1} = a_r^{-1}\ldots a_2^{-1}a_1^{-1}$ is called the *inverse* of A. If $\{a, b, a^{-1}, b^{-1}\}$ appear in a sequence with the form as $AaBbCa^{-1}Db^{-1}E$, then they are said to be an *interlaced set*; otherwise, a *parallel set*. Let $\widetilde{S}$ be the set of all surfaces. For a surface $S \in \widetilde{S}$, we obtain its genus $g(S)$ by using the following transforms to determine its equivalence to one of the standard forms.

Transform 1 $Aaa^{-1} \sim A$, where $A \in \widetilde{S}$ and $a \notin A$.

Transform 2 $AabBb^{-1}a^{-1} \sim AcBc^{-1}$.

Transform 3 $(Aa)(a^{-1}B) \sim (AB)$.

Transform 4 $AaBbCa^{-1}Db^{-1}E \sim ADCBEaba^{-1}b^{-1}$.

In the above transforms, the parentheses stand for cyclic order. For convenience, the parentheses are always omitted when unnecessary to distinguish cyclic or linear order. For more details concerning surfaces, the reader is referred to [10-11] and [14].

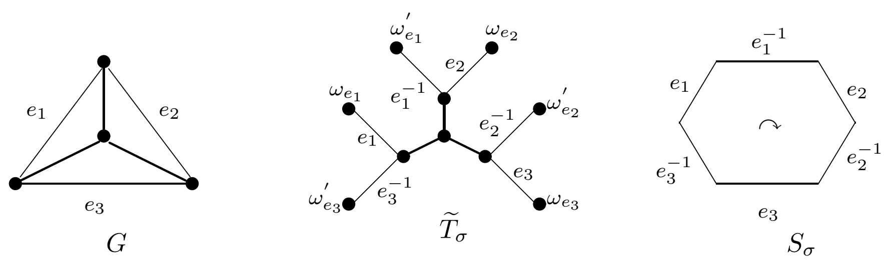


Fig. 1.

For a graphical property $\mathscr{P}$, a *Smarandache $\mathscr{P}$-drawing* of a graph G is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$ and *optimal* if $\mathscr{P} = G$ with minimized crossings. Let T be a spanning tree of a graph $G = (V, E)$, then $E = E_T + E_T^*$, where E_T consists of all the tree edges, and $E_T^* = \{e_1, e_2, \ldots e_\beta\}$ consists of all the co-tree edges, where $\beta = \beta(G)$ is the cycle rank of G. Split each co-tree edge $e_i = (\mu_{e_i}, \nu_{e_i}) \in E_T^*$ into two semi-edges $(\mu_{e_i}, \omega_{e_i})$, $(\nu_{e_i}, \omega'_{e_i})$, denoted by e_i^{+1} (or simply by e_i if no confusion) and e_i^{-1} respectively. Let $\widetilde{T} = (V + V_1, E + E_1)$, where $V_1 = \{\omega_{e_i},\ \omega'_{e_i} \mid 1 \leqslant i \leqslant \beta\}$, $E_1 = \{(\mu_{e_i}, \omega_{e_i}),\ (\nu_{e_i}, \omega'_{e_i}) \mid 1 \leqslant i \leqslant \beta\}$. Obviously, $\widetilde{T}$ is a tree. A *rotation at a vertex v*, which is denoted by σ_v, is a cyclic permutation of edges incident on v. A rotation system $\sigma = \sigma_G$ for a graph G is a set $\{\sigma_v | \forall v \in V(G)\}$. The tree $\widetilde{T}$ with a rotation system of G is called a *joint-tree* of G, and is denoted by $\widetilde{T}_\sigma$. Because it ia a tree, it can be embedded in the plane. By reading the lettered semi-edges of $\widetilde{T}_\sigma$ in a fixed direction (clockwise or anticlockwise), we can

get an algebraic representation of the surface which is represented by a $2\beta-$polygon. Such a surface, which is denoted by S_σ, is called an associated surface of $\widetilde{T}_\sigma$. A joint-tree $\widetilde{T}_\sigma$ of G and its associated surface is illustrated by Fig.1, where the rotation at each vertex of G complies with the clockwise rotation. From [10], there is 1-1 correspondence between associated surfaces (or joint-trees) and embeddings of a graph.

To *merge* a vertex of degree two is that replace its two incident edges with a single edge joining the other two incident vertices. *Vertex-splitting* is such an operation as follows. Let v be a vertex of graph G. We replace v by two new vertices v_1 and v_2. Each edge of G joining v to another vertex u is replaced by an edge joining u and v_1, or by an edge joining u and v_2. A graph is called a *cactus* if all circuits are independent, *i.e.*, pairwise vertex-disjoint. The *maximum genus* $\gamma_M(G)$ of a connected graph G is the maximum integer k such that there exists an embedding of G into the orientable surface of genus k. Since any embedding must have at least one face, the Euler characteristic for one face leads to an upper bound on the maximum genus

$$\gamma_M(G) \leq \lfloor \frac{|E(G)| - |V(G)| + 1}{2} \rfloor.$$

A graph G is said to be *upper embeddable* if $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, where $\beta(G) = |E(G)| - |V(G)| + 1$ denotes the *Betti number* of G. Obviously, the maximum genus of a cactus is zero. The vertex v of a graph G is called a 1-*critical-vertex* for the maximum genus of the graph, or for simplicity called 1-*critical-vertex*, if $G-v$ is a connected graph and $\gamma_M(G-v) = \gamma_M(G)-1$. Graphs considered here are all connected, undirected, and with minimum degree at least three. In addition, the surfaces are all orientable. Notations and terminologies not defined here can be seen in [1] and [9-11].

Lemma 1.0 *If there is a joint-tree $\widetilde{T}_\sigma$ of G such that the genus of its associated surface equals $\lfloor \beta(G)/2 \rfloor$ then G is upper embeddable.*

Proof According to the definition of joint-tree, associated surface, and upper embeddable graph, Lemma 1.0 can be easily obtained. □

Lemma 1.1 *Let AB be a surface. If $x \notin A \cup B$, then $g(AxBx^{-1}) = g(AB)$ or $g(AxBx^{-1}) = g(AB) + 1$.*

Proof First discuss the topological standard form of the surface AB.

(I) According to the left to right direction, let $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ be the first interlaced set appeared in A. Performing Transform 4 on $\{x_1, y_1, x_1^{-1}, y_1^{-1}\}$ we will get $A'Bx_1y_1x_1^{-1}y_1^{-1}$ ($\sim AB$). Then perform Transform 4 on the first interlaced set in A'. And so on. Eventually we will get $\widetilde{A}B\prod_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1}$ ($\sim AB$), where there is no interlaced set in $\widetilde{A}$.

(II) For the surface $\widetilde{A}B\prod_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1}$, from the left of B, successively perform Transform 4 on B similar to that on A in (I). Eventually we will get $\widetilde{A}\widetilde{B}\prod_{i=1}^{r} x_iy_ix_i^{-1}y_i^{-1}\prod_{j=1}^{s} a_jb_ja_j^{-1}b_j^{-1}$ ($\sim$

AB), where there is no interlaced set in $\widetilde{B}$.

(III) For the surface $\widetilde{A}\widetilde{B}\prod_{i=1}^{r} x_i y_i x_i^{-1} y_i^{-1} \prod_{j=1}^{s} a_j b_j a_j^{-1} b_j^{-1}$, from the left of $\widetilde{A}\widetilde{B}$, successively perform Transform 4 on $\widetilde{A}\widetilde{B}$ similar to that on A in (I). At last, we will get $\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}$, which is the topologically standard form of the surface AB.

As for the surface $AxBx^{-1}$, perform Transform 4 on A and B similar to that on A in (I) and B in (II) respectively. Eventually $\widetilde{A}x\widetilde{B}x^{-1}\prod_{i=1}^{r} x_i y_i x_i^{-1} y_i^{-1} \prod_{j=1}^{s} a_j b_j a_j^{-1} b_j^{-1}$ ($\sim AxBx^{-1}$) will be obtained. Then perform the same Transform 4 on $\widetilde{A}x\widetilde{B}x^{-1}$ as that on $\widetilde{A}\widetilde{B}$ in (III), and at last, one more Transform 4 than that in (III) may be needed because of x and x^{-1} in $\widetilde{A}x\widetilde{B}x^{-1}$. Eventually $\prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1}$ or $\prod_{i=1}^{p+1} a_i b_i a_i^{-1} b_i^{-1}$, which is the topologically standard form of the surface $AxBx^{-1}$, will be obtained.

From the above, Lemma 1.1 is obtained. □

Lemma 1.2 *Among all orientable surfaces represented by the linear sequence consisting of* a_i *and* a_i^{-1} $(i = 1, \ldots, n)$*, the surface* $a_1 a_2 \ldots a_n a_1^{-1} a_2^{-1} \ldots a_n^{-1}$ *is one whose genus is maximum.*

Proof According to Transform 4, Lemma 1.2 can be easily obtained. □

Lemma 1.3 *Let* G *be a graph with minimum degree at least three, and* $\bar{G}$ *be the graph obtained from* G *by a sequence of vertex-splitting, then* $\gamma_M(\bar{G}) \le \gamma_M(G)$*. Furthermore, if* $\bar{G}$ *is upper embeddable then* G *is upper embeddable as well.*

Proof Let v be a vertex of degree $n(\ge 4)$ in G, and G' be the graph obtained from G by splitting the vertex v into two vertices such that both their degrees are at least three. First of all, we prove that the maximum genus will not increase after one vertex-splitting operation, *i.e.*, $\gamma_M(G') \le \gamma_M(G)$.

Let $e_1, e_2, \ldots e_n$ be the n edges incident to v, and v be split into v_1 and v_2. Without loss of generality, let $e_{i_1}, e_{i_2}, \ldots e_{i_r}$ be incident to v_1, and $e_{i_{r+1}}, \ldots e_{i_n}$ be incident to v_2, where $2 \le i_r \le n-2$. Select such a spanning tree T of G that e_{i_1} is a tree edge, and $e_{i_2}, \ldots e_{i_n}$ are all co-tree edges. As for graph G', select T^* be a spanning tree such that both e_{i_1} and (v_1, v_2) are tree edges, and the other edges of T^* are the same as the edges in T. Obviously, $e_{i_2}, \ldots e_{i_n}$ are co-tree edges of T^*. Let $\mathcal{T}=\{\hat{T}_\sigma | \hat{T}_\sigma = \overline{(T-v)}_\sigma$, where $\overline{(T-v)}_\sigma$ is a joint-tree of $G - v\}$, $\mathcal{T}^*=\{\hat{T}^*_\sigma | \hat{T}^*_\sigma = \overline{(T^* - \{v_1, v_2\})}_\sigma$, where $\overline{(T^* - \{v_1, v_2\})}_\sigma$ is a joint-tree of $G' - \{v_1, v_2\}\}$. It is obvious that $\mathcal{T} = \mathcal{T}^*$. Let $\mathcal{S}$ be the set of all the associated surfaces of the joint-trees of G, and $\mathcal{S}^*$ be the set of all the associated surfaces of the joint trees of G'. Obviously, $\mathcal{S}^* \subseteq \mathcal{S}$. Furthermore, $|\mathcal{S}^*| = r! \times (n-r)! \times |\mathcal{T}^*| < |\mathcal{S}| = (n-1)! \times |\mathcal{T}|$. So $\mathcal{S}^* \subset \mathcal{S}$, and we have $\gamma_M(G') \le \gamma_M(G)$.

Reiterating this procedure, we can get that $\gamma_M(\bar{G}) \le \gamma_M(G)$. Furthermore, because $\beta(G) = \beta(\bar{G})$, it can be obtained that if $\bar{G}$ is upper embeddable then $\lfloor \frac{\beta(G)}{2} \rfloor = \lfloor \frac{\beta(\bar{G})}{2} \rfloor = \gamma_M(\bar{G}) \le \gamma_M(G) \le \lfloor \frac{\beta(G)}{2} \rfloor$. So, $\gamma_M(G) = \lfloor \frac{\beta(G)}{2} \rfloor$, and G is upper embeddable. □

§2. Results Related to 1-Critical-Vertex

The *neckband* $\mathcal{N}_{2n}$ is such a graph that $\mathcal{N}_{2n} = C_{2n} + R$, where C_{2n} is a 2n-cycle, and $R = \{a_i | a_i = (v_{2i-1}, v_{2i+2}).\ (i = 1, 2, \ldots, n,\ 2i + 2 \equiv r(mod\ 2n),\ 1 \leq r < 2n)\}$. The *möbius ladder* $\mathcal{M}_{2n}$ is such a cubic circulant graph with 2n vertices, formed from a 2n-cycle by adding edges (called "rungs") connecting opposite pairs of vertices in the cycle. For example, Fig. 2.1 and Fig. 2.5 is a graph of $\mathcal{N}_8$ and $\mathcal{M}_{2n}$ respectively. A vertex like the solid vertex in Fig. 2.2, Fig. 2.3, Fig. 2.4, Fig. 2.5, and Fig. 2.6 is called an *α-vertex*, *β-vertex*, *γ-vertex*, *δ-vertex*, and *η-vertex* respectively, where Fig. 2.6 is a neckband.

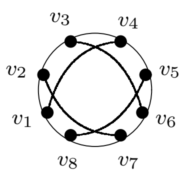


Fig. 2.1.

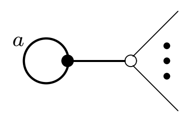


Fig. 2.2

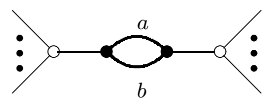


Fig. 2.3

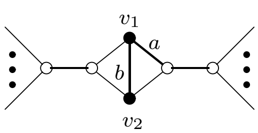


Fig. 2.4

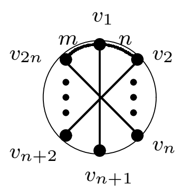


Fig. 2.5

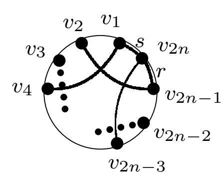


Fig. 2.6

Theorem 2.1 *If v is an α-vertex of a graph G, then $\gamma_M(G-v) = \gamma_M(G)$. If v is a β-vertex, or a γ-vertex, or a δ-vertex, or an η-vertex of a graph G, and $G-v$ is a connected graph, then $\gamma_M(G-v) = \gamma_M(G)-1$, i.e., β-vertex, γ-vertex, δ-vertex and η-vertex are 1-critical-vertex.*

Proof If v is an *α-vertex* of the graph G, then it is easy to get that $\gamma_M(G-v) = \gamma_M(G)$. In the following, we will discuss the other cases.

Case 1 v is an *β-vertex* of G.

According to Fig. 2.3, select such a spanning tree T of G such that both a and b are co-tree edges. It is obvious that the associated surface for each joint-tree of G must be one of the following four forms:

(i) $AabBa^{-1}b^{-1} \sim ABaba^{-1}b^{-1}$;
(ii) $AabBb^{-1}a^{-1} \sim AcBc^{-1}$;
(iii) $AbaBa^{-1}b^{-1} \sim AcBc^{-1}$;
(iv) $AbaBb^{-1}a^{-1} \sim ABbab^{-1}a^{-1}$.

On the other hand, for each joint-tree $\widetilde{T}^*_\sigma$, which is a joint-tree of $G-v$, its associated surface must be the form as AB, where A and B are the same as that in the above four forms.

According to (i)-(iv), Lemma 1.1, and $g(ABaba^{-1}b^{-1})$=$g(AB)+1$, we can get that $\gamma_M(G-v)$ $=\gamma_M(G)-1$.

Case 2 v is an γ-*vertex* of G.

As illustrated by Fig.2.4, both v_1 and v_2 are γ-*vertex*. Without loss of generality, we only prove that $\gamma_M(G-v_1)=\gamma_M(G)-1$. Select such a spanning tree T of G such that both a and b are co-tree edges. The associated surface for each joint-tree of G must be one of the following 16 forms:

$$\begin{array}{llll} Aabb^{-1}a^{-1}B, & Aabb^{-1}Ba^{-1}, & Aaba^{-1}Bb^{-1}, & AabBa^{-1}b^{-1}, \\ Abab^{-1}a^{-1}B, & Abab^{-1}Ba^{-1}, & Abaa^{-1}Bb^{-1}, & AbaBa^{-1}b^{-1}, \\ Ab^{-1}a^{-1}Bab, & Ab^{-1}Ba^{-1}ab, & Aa^{-1}Bb^{-1}ab, & ABa^{-1}b^{-1}ab, \\ Ab^{-1}a^{-1}Bba, & Ab^{-1}Ba^{-1}ba, & Aa^{-1}Bb^{-1}ba, & ABa^{-1}b^{-1}ba. \end{array}$$

Furthermore, each of these 16 types of surfaces is topologically equivalent to one of such surfaces as AB, $ABaba^{-1}b^{-1}$, and $AcBc^{-1}$. On the other hand, for each joint-tree $\widetilde{T}^*_\sigma$, which is a joint-tree of $G-v_1$, its associated surface must be the form of AB, where A and B are the same as that in the above 16 forms. According to Lemma 1.1 and $g(ABaba^{-1}b^{-1})$=$g(AB)+1$, we can get that $\gamma_M(G-v)=\gamma_M(G)-1$.

Case 3 v is an δ-*vertex* of G.

In Fig.2.5, let $a_i=(v_i,v_{n+i}), i=1,2,\ldots,n$. Without loss of generality, we only prove that $\gamma_M(G-v_1)=\gamma_M(G)-1$. Select such a joint-tree $\widetilde{T}_\sigma$ of Fig. 2.5, which is illustrated by Fig.3, where the edges of the spanning tree are represented by solid line. It is obvious that the associated surface of $\widetilde{T}_\sigma$ is $mnm^{-1}n^{-1}a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1}$. On the other hand, $a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1}$ is the associated surface of one of the joint-trees of $G-v_1$. From Lemma 1.2 and $g(mnm^{-1}n^{-1}a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1})$=$g(a_2a_3\ldots a_na_2^{-1}a_3^{-1}\ldots a_n^{-1})+1$, we can get that $\gamma_M(G-v)=\gamma_M(G)-1$.

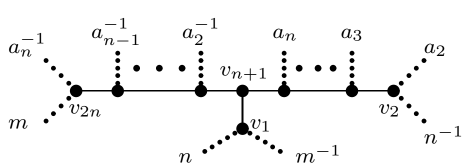


Fig. 3.

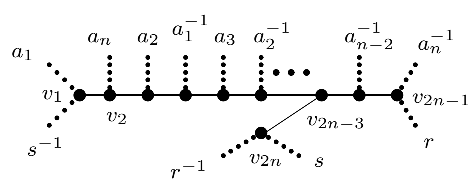


Fig. 4.

Case 4 v is an η-*vertex* of G.

As illustrated by Fig.2.6, every vertex in Fig. 2.6 is a η-*vertex*. Without loss of generality, we only prove that $\gamma_M(G-v_{2n})=\gamma_M(G)-1$.

A joint-tree $\widetilde{T}_\sigma$ of Fig.2.6 is depicted by Fig.4. It can be read from Fig.4 that the associated surface of $\widetilde{T}_\sigma$ is $S=a_1a_n(\prod\limits_{i=1}^{n-3}a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}rsr^{-1}s^{-1}$. Performing a sequence of Transform

4 on S, we have

$$\begin{aligned}
S &= a_1a_n(\prod_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}rsr^{-1}s^{-1}\\
\text{(Transform 4)} &\sim (\prod_{i=2}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_2rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}\\
\text{(Transform 4)} &\sim (\prod_{i=4}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_4rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}a_3a_2a_3^{-1}a_2^{-1}\\
\cdots & \quad \cdots\\
\text{(Transform 4)} &\sim \begin{cases} rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}(\prod\limits_{i=2}^{n-4} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n\equiv 0(mod\ 2);\\ rsr^{-1}s^{-1}a_1a_na_1^{-1}a_n^{-1}(\prod\limits_{i=2}^{n-3} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n\equiv 1(mod\ 2).\end{cases}
\end{aligned} \tag{1}$$

It is known from (1) that

$$g(S)=\gamma_M(G) \tag{2}$$

On the other hand, $S^{'}=a_1a_n(\prod\limits_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}$ is the associated surface of $\widetilde{T}_\sigma^*$, where $\widetilde{T}_\sigma^*$ is a joint-tree of $G-v_{2n}$. Performing a sequence of Transform 4 on $S^{'}$, we have

$$\begin{aligned}
S^{'} &= a_1a_n(\prod_{i=1}^{n-3} a_{i+1}a_i^{-1})a_{n-2}^{-1}a_n^{-1}\\
&\sim \begin{cases} a_1a_na_1^{-1}a_n^{-1}(\prod\limits_{i=2}^{n-4} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n\equiv 0(mod\ 2);\\ a_1a_na_1^{-1}a_n^{-1}(\prod\limits_{i=2}^{n-3} a_{i+1}a_ia_{i+1}^{-1}a_i^{-1}) & n\equiv 1(mod\ 2).\end{cases}
\end{aligned} \tag{3}$$

It can be inferred from (3) that

$$g(S^{'})=\gamma_M(G-v_{2n}). \tag{4}$$

From (1) and (3) we have

$$g(S)=g(S^{'})+1. \tag{5}$$

From (2), (4), and (5) we have $\gamma_M(G-v_{2n})=\gamma_M(G)-1$.

According to the above, we can get Theorem 2.1. □

Let G be a connected graph with minimum degree at least 3. The following algorithm can be used to get the maximum genus of G.

Algorithm I:

Step 1 Input $i=0$, $G_0=G$.

Step 2 If there is a 1-*critical-vertex* v in G_i, then delete v from G_i and go to Step 3. Else, go to Step 4.

Step 3 Deleting all the vertices of degree one and merging all the vertices of degree two in $G_i - v$, we get a new graph G_{i+1}. Let $i = i + 1$, then go back to Step 2.

Step 4 Output $\gamma_M(G) = \gamma_M(G_i) + i$.

Remark Using Algorithm I, the computing of the maximum genus of G can be reduced to the computing of the maximum genus of G_i, which may be much easier than that of G.

§3. Upper Embeddability of Graphs

An *ear* of a graph G, which is the same as the definition offered in [16], is a path that is maximal with respect to internal vertices having degree 2 in G and is contained in a cycle in G. An *ear decomposition* of G is a decomposition $p_0, \ldots, p_k$ such that p_0 is a cycle and p_i for $i \geqslant 1$ is an ear of $p_0 \cup \cdots \cup p_i$. A *spiral* $\mathcal{S}_m^n$ is the graph which has an ear decomposition $p_0, \ldots, p_n$ such that p_0 is the m-cycle $(v_1 v_2 \ldots v_m)$, p_i for $1 \leqslant i \leqslant m-1$ is the 3-path $v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_i$ which joining v_{m+2i-2} and v_i, and p_i for $i > m-1$ is the 3-path $v_{m+2i-2}v_{m+2i-1}v_{m+2i}v_{2i-m+1}$ which joining v_{m+2i-2} and v_{2i-m+1}. If some edges in $\mathcal{S}_m^n$ are replaced by the graph depicted by Fig. 6, then the graph is called an *extended-spiral*, and is denoted by $\tilde{\mathcal{S}}_m^n$. Obviously, both the vertex v_1 and v_2 in Fig. 6 are γ-vertex. For convenience, a graph of $\mathcal{S}_5^6$ is illustrated by Fig.5, and Fig.7 is the graph which is obtained from $\mathcal{S}_5^6$ by replacing the edge (v_{13}, v_{14}) with the graph depicted by Fig.6.

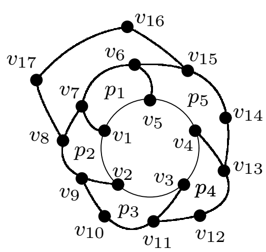


Fig. 5.

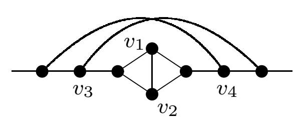


Fig. 6.

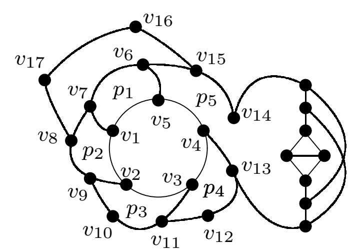


Fig. 7.

Theorem 3.1 *The graph $\mathcal{S}_5^n$ is upper embeddable. Furthermore, $\gamma_M(\mathcal{S}_5^n - v_{2n+3}) = \gamma_M(\mathcal{S}_5^n) - 1$, i.e., v_{2n+3} is a 1-critical-vertex of $\mathcal{S}_5^n$.*

Proof According to the definition of $\mathcal{S}_5^n$, when $n \leq 4$, it is not a hard work to get the upper embeddability of $\mathcal{S}_5^n$. So the following 5 cases will be considered.

Case 1 $n = 5j$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2$, where T_1 is the path $v_2v_1v_5v_4v_3\{\prod_{i=1}^{j-1} v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}\}v_{2n+1}$- v_{2n} $v_{2n-1}v_{2n-2}v_{2n-3}v_{2n-4}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. Obviously, the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_2, v_3)$, $e_2 = (v_2, v_9)$, $e_3 = (v_1, v_7)$, $\prod_{i=1}^{j-1}\{e_{5i-1} = (v_{10i-5}, v_{10i-4}), e_{5i} = (v_{10i-6}, v_{10i+3}), e_{5i+1} = (v_{10i+1}, v_{10i+2}), e_{5i+2} = (v_{10i}, v_{10i+9}), e_{5i+3} =$

$(v_{10i-2}, v_{10i+7})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Select such a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$ which is depicted by Fig.8. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_1^{-1}e_3e_4e_5\{\prod_{i=1}^{j-2} e_{5i+1}e_{5i+2}e_{5i-3}^{-1}e_{5i+3}e_{5i-2}^{-1}e_{5i-1}^{-1}e_{5i+4}e_{5i+5}e_{5i}^{-1}e_{5i+1}^{-1}\} \\ & \quad e_{n-4}e_{n-3}e_{n-8}^{-1}e_{n-2}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\dfrac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\dfrac{n+1}{2}\rfloor$. So, when $n = 5j$, $\mathcal{S}_5^n$ is upper embeddable.

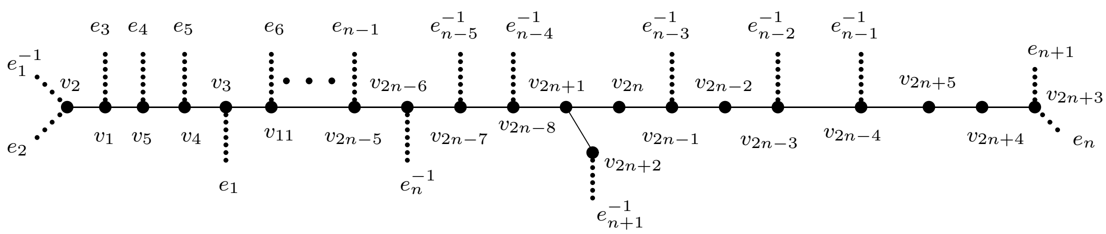


Fig. 8.

Case 2 $n = 5j + 1$, where j is an integer no less than 1.

Without loss of generality, select $T = T_1 \cup T_2$ to be a spanning tree of $\mathcal{S}_5^n$, where T_1 is the path $v_3v_2v_1\{\prod_{i=1}^{j} v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_3, v_4)$, $e_3 = (v_3, v_{11})$, $e_4 = (v_2, v_9)$, $\prod_{i=1}^{j-1}\{e_{5i} = (v_{10i-3}, v_{10i-2}), e_{5i+1} = (v_{10i-4}, v_{10i+5}), e_{5i+2} = (v_{10i+3}, v_{10i+4}), e_{5i+3} = (v_{10i+2}, v_{10i+11}), e_{5i+4} = (v_{10i}, v_{10i+9})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_6e_1^{-1}e_2^{-1}e_7e_8e_3^{-1}e_9\{\prod_{i=1}^{j-2} e_{5i-1}^{-1}e_{5i}^{-1}e_{5i+5}e_{5i+6}e_{5i+1}^{-1}e_{5i+2}^{-1}e_{5i+7} \\ & \quad e_{5i+8}e_{5i+3}^{-1}e_{5i+9}\}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\dfrac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\dfrac{n+1}{2}\rfloor$. So, when $n = 5j + 1$, $\mathcal{S}_5^n$ is upper embeddable.

Case 3 $n = 5j + 2$, where j is an integer no less than 1.

Without loss of generality, select a spanning tree of $\mathcal{S}_5^n$ to be $T = T_1 \cup T_2$, where T_1 is the path $v_1v_5v_4v_3v_2\{\prod_{i=1}^{j} v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i+5}v_{10i+4}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_2)$, $e_2 = (v_1, v_7)$, $e_3 = (v_5, v_6)$, $e_4 = (v_4, v_{13})$, $e_5 = (v_3, v_{11})$, $\prod_{i=1}^{j-1}\{e_{5i+1} = (v_{10i-1}, v_{10i}), e_{5i+2} = (v_{10i-2}, v_{10i+7}), e_{5i+3} = (v_{10i+5}, v_{10i+6}), e_{5i+4} = (v_{10i+4}, v_{10i+13}), e_{5i+5} = (v_{10i+2}, v_{10i+11})\}$, $e_{n-1} = (v_{2n-5}, v_{2n-4})$, $e_n = (v_{2n-6}, v_{2n+3})$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_1^{-1}e_6e_7e_2^{-1}e_3^{-1}e_8e_9e_4^{-1}e_{10}\{\prod_{i=1}^{j-2} e_{5i}^{-1}e_{5i+1}^{-1}e_{5i+6}e_{5i+7}e_{5i+2}^{-1}e_{5i+3}^{-1}e_{5i+8} \\ &\quad e_{5i+9}e_{5i+4}^{-1}e_{5i+10}\}e_{n-7}^{-1}e_{n-6}^{-1}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j+2$, $\mathcal{S}_5^n$ is upper embeddable.

Case 4 $n = 5j + 3$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2$, where T_1 is the path $v_2v_1v_7v_6v_5v_4v_3\{\prod_{i=1}^{j} v_{10i+1}v_{10i}v_{10i-1}v_{10i-2}v_{10i+7}v_{10i+6}v_{10i+5}v_{10i+4}v_{10i+3}$-$v_{10i+2}\}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_2, v_3)$, $e_3 = (v_2, v_9)$, $\prod_{i=1}^{j}\{e_{5i-1} = (v_{10i-3}, v_{10i-2}), e_{5i} = (v_{10i-4}, v_{10i+5}), e_{5i+1} = (v_{10i-6}, v_{10i+3}), e_{5i+2} = (v_{10i+1}, v_{10i+2}), e_{5i+3} = (v_{10i}, v_{10i+9})\}$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_1e_2e_3e_4e_5e_1^{-1}e_6e_2^{-1}\{\prod_{i=1}^{j-1} e_{5i+2}e_{5i+3}e_{5i-2}^{-1}e_{5i-1}^{-1}e_{5i+4}e_{5i+5}e_{5i}^{-1}e_{5i+6} \\ &\quad e_{5i+1}^{-1}e_{5i+2}^{-1}\}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor \frac{n+1}{2} \rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor \frac{n+1}{2} \rfloor$. So, when $n = 5j+3$, $\mathcal{S}_5^n$ is upper embeddable.

Case 5 $n = 5j + 4$, where j is an integer no less than 1.

Without loss of generality, a spanning tree T of $\mathcal{S}_5^n$ can be chosen as $T = T_1 \cup T_2 \cup T_3$, where T_1 is the path $v_1v_2\{\prod_{i=1}^{j} v_{10i-1}v_{10i-2}v_{10i-3}v_{10i-4}v_{10i-5}v_{10i-6}v_{10i+3}v_{10i+2}v_{10i+1}v_{10i}\}v_{2n+1}$-$v_{2n}v_{2n-1}v_{2n-2}v_{2n-3}v_{2n-4}v_{2n+5}v_{2n+4}v_{2n+3}$, $T_2 = (v_2, v_3)$, $T_3 = (v_{2n+1}, v_{2n+2})$. It is obviously that the $n+1$ co-tree edges of $\mathcal{S}_5^n$ with respect to T are $e_1 = (v_1, v_5)$, $e_2 = (v_1, v_7)$, $e_3 = (v_3, v_4)$, $e_4 = (v_3, v_{11})$, $\prod_{i=1}^{j}\{e_{5i} = (v_{10i-1}, v_{10i}), e_{5i+1} = (v_{10i-2}, v_{10i+7}), e_{5i+2} = (v_{10i-4}, v_{10i+5}), e_{5i+3} = (v_{10i+3}, v_{10i+4}), e_{5i+4} = (v_{10i+2}, v_{10i+11})\}$, $e_{n+1} = (v_{2n+2}, v_{2n+3})$. Similar to Case 1, select a joint-tree $\widetilde{T}_\sigma$ of $\mathcal{S}_5^n$. After a sequence of Transform 4, the associated surface S of $\widetilde{T}_\sigma$ has the form as

$$\begin{aligned} S &= e_2e_1e_3e_4e_5e_6e_2^{-1}e_7e_1^{-1}e_3^{-1}\{\prod_{i=1}^{j-1} e_{5i+3}e_{5i+4}e_{5i-1}^{-1}e_{5i}^{-1}e_{5i+5}e_{5i+6}e_{5i+1}^{-1} \\ &\quad e_{5i+7}e_{5i+2}^{-1}e_{5i+3}^{-1}\}e_{n-1}e_{n-5}^{-1}e_{n-4}^{-1}e_{n-3}^{-1}e_{n-2}^{-1}e_{n-1}^{-1}e_{n+1}e_ne_{n+1}^{-1}e_n^{-1} \\ &\sim \prod_{i=1}^{\lfloor\frac{n+1}{2}\rfloor} e_{i1}e_{i2}e_{i1}^{-1}e_{i2}^{-1}, \end{aligned}$$

where e_{ij}, $e_{ij}^{-1} \in \{e_1, \ldots, e_{n+1}, e_1^{-1}, \ldots, e_{n+1}^{-1}\}$; $i = 1, \ldots, \lfloor\frac{n+1}{2}\rfloor$; $j = 1, 2$. Obviously, $g(S) = \lfloor\frac{n+1}{2}\rfloor$. So, when $n = 5j+4$, $\mathcal{S}_5^n$ is upper embeddable.

From the Case 1-5, the upper embeddability of $\mathcal{S}_5^n$ can be obtained.

Similar to the Case 1-5, for each $n \geq 5$, there exists a joint-tree $\widetilde{T}_\sigma^*$ of $\mathcal{S}_5^n - v_{2n+3}$ such that its associated surface is $S' = S - \{e_{n+1}e_ne_{n+1}^{-1}e_n^{-1}\}$. It is obvious that S' is the surface into which the embedding of $\mathcal{S}_5^n - v_{2n+3}$ is the maximum genus embedding. Furthermore, $g(S') = g(S) - 1$, *i.e.*, $\gamma_M(\mathcal{S}_5^n - v_{2n+3}) = \gamma_M(\mathcal{S}_5^n) - 1$. So, v_{2n+3} is a 1-*critical-vertex* of $\mathcal{S}_5^n$. □

Similar to the proof of Theorem 3.1, we can get the following conclusions.

Theorem 3.2 *The graph $\mathcal{S}_m^n$ is upper embeddable. Furthermore, $\gamma_M(\mathcal{S}_m^n - v_{m+2n-2}) = \gamma_M(\mathcal{S}_m^n) - 1$, i.e., v_{m+2n-2} is a 1-critical-vertex of $\mathcal{S}_m^n$.*

Corollary 3.1 *Let G be a graph with minimum degree at least three. If G, through a sequence of vertex-splitting operations, can be turned into a spiral $\mathcal{S}_m^n$, then G is upper embeddable.*

Proof According to Lemma 1.3, Theorem 3.2, and the upper embeddability of graphs, Corollary 3.1 can be obtained. □

In the following, we will offer an algorithm to obtain the maximum genus of the *extended-spiral* $\tilde{\mathcal{S}}_m^n$.

Algorithm II:

Step 1 Input $i = 0$ and $j = 0$. Let G_0 be the *extended-spiral* $\tilde{\mathcal{S}}_m^n$.

Step 2 If there is a γ-vertex v in G_i, then delete v from G_i, and go to Step 3. Else, go to Step 4.

Step 3 Deleting all the vertices of degree one and merging some vertices of degree two in $G_i - v$, we get a new graph G_{i+1}. Let $i = i + 1$. If G_i is a *spiral* $\mathcal{S}_m^n$, then go to Step 4. Else,

go back to Step 2.

Step 4 Let G_{i+j} be the *spiral* $\mathcal{S}_m^n$. Deleting v_{m+2n-2} from $\mathcal{S}_m^n$, we will get a new graph G_{i+j+1}, (obviously, G_{i+j+1} is either a *spiral* $\mathcal{S}_m^{n-2}$ or a *cactus*).

Step 5 If G_{i+j+1} is a *cactus*, then go to Step 6. Else, Let $n = n - 2$, $j = j + 1$ and go back to Step 4.

Step 6 Output $\gamma_M(\tilde{\mathcal{S}}_m^n) = i + j + 1$.

Remark 1. In the graph G depicted by Fig.6, after deleting a γ-vertex v_1 (or v_2) from G, the vertex v_3 (or v_4) is still a γ-vertex of the remaining graph.

2. From Algorithm II we can get that the *extended-spiral* $\tilde{\mathcal{S}}_m^n$ is upper embeddable.

References

[1] Bondy J.A., Murty U.S.R., *Graph Theory with Applications*, Macmillan, London, 1976.

[2] Cai J., Dong G. and Liu Y., A suffcient condition on upper embeddability of graphs, *Science China Mathematics*, 53(5), 1377-1384 (2010).

[3] Chen J., Kanchi S. P. and Gross J. L., A tight lower bound on the maximum genus of a simplicial graph, *Discrete Math.*, 156, 83-102 (1996)

[4] Chen Y., Liu Y., Upper embeddability of a graph by order and girth, *Graphs and Combinatorics*, 23, 521-527 (2007)

[5] Hao R., Xu L., ect., Embeddable properties of digraphs in orientable surfaces, *Acta Mathematicae Applicatae Sinica* (Chinese Ser.), 31(4), 630 -634 (2008).

[6] Huang Y., Liu Y., Face size and the maximum genus of a graph, *J Combin. Theory Ser B.*, 80, 356–370 (2000)

[7] Li Z., Ren H., Maximum genus embeddings and minimum genus embeddings in non-orientable surfaces, *Acta Mathematica Sinica*, Chinese Series, 54(2), 329-332, (2011).

[8] Liu Y., The maximum orientable genus of a graph, *Scientia Sinical (Special Issue)*, **(II)**, 41-55 (1979)

[9] Liu Y., *Embeddability in Graphs*, Kluwer Academic, Dordrecht, Boston, London, 1995.

[10] Liu Y., *Theory of Polyhedra*, Science Press, Beijing, 2008.

[11] Liu Y., *Topological Theory on Graphs*, USTC Press, Hefei, 2008.

[12] Nordhause E.A., Stewart B.M., White A.T., On the maximum genus of a graph, *J. Combin. Theory*, 11, 258-267 (1971).

[13] Ren H., Li G., Survey of maximum genus of graphs, *J. East China Normal University*(Natural Sc), 5: 1-13 (2010).

[14] Ringel G., *Map Color Theorem*, Springer, 1974.

[15] Škoviera M., The maximum genus of graphs diameter two, *Discrete Math.*, 87, 175−180 (1991)

[16] West D.B., *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, NJ, 2001.

[17] Xuong N.H., How to determine the maximum genus of a graph, *J. Combin. Theory* Ser. B., 26 217-225 (1979)

Math.Combin.Book Ser. Vol.4(2012), 92-96

Total Dominator Colorings in Cycles

A.Vijayalekshmi

(S.T.Hindu College, Nagercoil, Tamil Nadu-629 002, India)

E-mail: vijimath.a@gmail.com

Abstract: Let G be a graph without isolated vertices. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper we determine the total dominator chromatic number in cycles.

Key Words: Total domination number, chromatic number and total dominator chromatic number, Smarandachely k-dominator coloring, Smarandachely k-dominator chromatic number.

AMS(2010): 05C15, 05C69

§1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3]. Let $G = (V, E)$ be a graph of order n with minimum degree at least one. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\underset{v \in S}{\cup} N(v)$, and the closed neighborhood of S is $N[S] = N(S) \cup S$.

A subset S of V is called a total dominating set if every vertex in V is adjacent to some vertex in S. A total dominating set is minimal total dominating set if no proper subset of S is a total dominating set of G. The total domination number γ_t is the minimum cardinality taken over all minimal total dominating sets of G. A γ_t-set is any minimal total dominating set with cardinality γ_t.

A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$. Let $V = \{u_1, u_2, u_3, \ldots, u_p\}$ and $\c{C} = \{C_1, C_2, C_3, \ldots, C_n\}, n \leqslant p$ be a collection of subsets $C_i \subset V$. A color represented in a vertex u is called a non-repeated color if there exists one color class $C_i \in \c{C}$ such that $C_i = \{u\}$.

Let G be a graph without isolated vertices. For an integer $k \geqslant 1$, a Smarandachely k-dominator coloring of G is a proper coloring of G with the extra property that every vertex

[1]Received September 2, 2012. Accepted December 16, 2012.

in G properly dominates a k-color classes and the smallest number of colors for which there exists a Smarandachely k-dominator coloring of G is called the Smarandachely k-dominator chromatic number of G and is denoted by $\chi_{td}^S(G)$. A total dominator coloring of a graph G is a proper coloring of G with the extra property that every vertex in G properly dominates a color class. The smallest number of colors for which there exists a total dominator coloring of G is called the total dominator chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper, we determine total dominator chromatic number in cycles.

Throughout this paper, we use the following notations.

Notation 1.1 *Usually, the vertices of C_n are denoted by $u_1, u_2, \ldots, u_n$ in order. For $i < j$, we use the notation $\langle [i,j] \rangle$ for the subpath induced by $\{u_i, u_{i+1}, \ldots, u_j\}$. For a given coloring C of C_n, $C|\langle [i,j] \rangle$ refers to the coloring C restricted to $\langle [i,j] \rangle$.*

We have the following theorem from [1].

Theorem 1.2([1]) *Let G be any graph with $\delta(G) \geq 1$. Then*

$$\max\{\chi(G), \gamma_t(G)\} \leq \chi_{td}(G) \leq \chi(G) + \gamma_t(G).$$

Definition 1.3 *We know from Theorem (1.2) that $\chi_{td}(P_n) \in \{\gamma_t(P_n), \gamma_t(P_n)+1, \gamma_t(P_n)+2\}$. We call the integer n, good (respectively bad, very bad) if $\chi_{td}(P_n) = \gamma_t(P_n) + 2$ (if respectively $\chi_{td}(P_n) = \gamma_t(P_n) + 1$, $\chi_{td}(P_n) = \gamma_t(P_n)$).*

First, we prove a result which shows that for large values of n, the behavior of $\chi_{td}(P_n)$ depends only on the residue class of $n mod 4$ [More precisely, if n is good, $m > n$ and $m \equiv n (\bmod 4)$ then m is also good]. We then show that $n = 8, 13, 15, 22$ are the least good integers in their respective residue classes. This therefore classifies the good integers.

Fact 1.4 Let $1 < i < n$ and let C be a td-coloring of P_n. Then, if either u_i has a repeated color or u_{i+2} has a non-repeated color, $C|\langle [i+1, n] \rangle$ is also a td-coloring. This fact is used extensively in this paper.

§2. Determination of $\chi_{td}(C_n)$

It is trivially true that $\chi_{td}(C_3) = 3$ and $\chi_{td}(C_4) = 2$. We assume $n \geq 5$.

Lemma 2.1 *If P_n has a minimum td-coloring in which the end vertices have different colors, then $\chi_{td}(C_n) \leq \chi_{td}(P_n)$.*

Proof Join $u_1 u_n$ by an edge and we get an induced td-coloring of C_n. □

Corollary 2.2 $\chi_{td}(C_n) \leq \chi_{td}(P_n)$ *for* $\forall n \neq 3, 11, 18$.

Lemma 2.3 *If C_n has a minimal td-coloring in which either there exists a color class of the form $N(x)$, where x is a non-repeated color or no color class of the form $N(x)$, then*

$\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Proof We have assumed $n > 3$. If $n = 3$, conclusion is trivially true. We have the following two cases.

Case 1 C_n has a minimal td-coloring C in which there is a color class of the form $N(x)$, where x is a non-repeated color. Let C_n be the cycle $u_1u_2 \ldots u_nu_1$. Let us assume $x = u_2$ has a non-repeated color n_1 and $N(x) = \{u_1, u_3\}$ is the color class of color r_1. Then u_{n-1} has a non-repeated color since u_n has to dominate a color class which must be contained in $N(u_n) = \{u_1, u_{n-1}\}$. Thus $C|\langle[1, n]\rangle$ is a td-coloring. Thus $\chi_{td}(P_n) \leq \chi_{td}(C_n)$.

Case 2 There exists C_n has a minimal td-coloring which has no color class of the form $N(x)$. It is clear from the assumption that any vertex with a non-repeated color has an adjacent vertex with non-repeated color. We consider two sub cases.

Subcase a There are two adjacent vertices u, v with repeated color. Then the two vertices on either side of u, v say u_1 and v_1 must have non-repeated colors. Then the removal of the edge uv leaves a path P_n and $C|\langle[1, n]\rangle$ is a td-coloring.

Subcase b There are adjacent vertices u, v with u (respectively v) having repeated (respectively non-repeated) color. Then consider the vertex $u_1(\neq v)$ adjacent to u. We may assume u_1 has non-repeated color (because of sub case (a)). v_1 must also have a non-repeated color since v must dominate a color class and u has a repeated color. Once again, $C|(C_n - uv)$ is a td-coloring and the proof is as in sub case (a). Since either sub case (a) or sub case (b) must hold, the lemma follows. □

Lemma 2.4 $\chi_{td}(C_n) = \chi_{td}(P_n)$ *for* $n = 8, 13, 15, 22$.

proof We prove for $n = 22$. By Lemma 2.1, $\chi_{td}(P_{22}) \geq \chi_{td}(C_{22})$. Let $\chi_{td}(C_{22}) < \chi_{td}(P_{22}) = 14$. Then by Lemma 2.3, C_{22} has a minimal td-coloring in which there is a color class of the form $N(x)$, where x is a repeated color (say C_1). Suppose $x = u_2$ First, we assume that the color class of u_2 is not $N(u_1)$ or $N(u_3)$. Then we have u_4, u_5, u_{22}, u_{21} must be non-repeated colors.

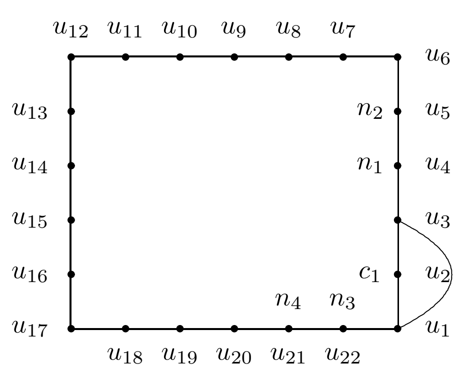


Fig.1

Then $C|\langle[6,20]\rangle$ is a coloring (which may not be a td-coloring for the section) with 8 colors including $C_1 \Rightarrow$ The vertices u_7 and u_{19} have the color C_1. (The sets $\{u_6, u_8\}, \{u_7, u_9\}, \{u_{10}, u_{12}\}$, $\{u_{11}, u_{13}\}, \{u_{14}, u_{16}\}, \{u_{15}, u_{17}\}, \{u_{18}, u_{20}\}$ must contain color classes. Therefore the remaining vertex u_{19} must have color C_1. Similarly, going the other way, we get u_7 must have color C_1). Then $\{u_6, u_8\}, \{u_{18}, u_{20}\}$ are color classes and $u_9, u_{10}, u_{16}, u_{17}$ are non-repeated colors. This leads $\langle[11, 15]\rangle$ to be colored with 2 colors including C_1, which is not possible. Hence $\chi_{td}(C_{22}) = 14 = \chi_{td}(P_{22})$. If the color class of u_2 is $N(u_1)$ or $N(u_3)$, the argument is similar. Proof is similar for n = 8,13,15. □

Lemma 2.5 *Let n be a good integer. Then* $\chi_{td}(P_n) \leq \chi_{td}(C_n)$

Proof We use induction on n. Let $u_1, u_2, \ldots, u_n$ be vertices of C_n in order. Let C be a minimal td-coloring of C_n. For the least good integers in their respective residue classes mod 4 is $8, 13, 15, 22$, the result is proved in the previous Lemma 2.4. So we may assume that the result holds for all good integers $< n$ and that $n-4$ is also a good integer. First suppose, there exists a color class of the form $N(x)$. Let $x = u_2$. Suppose u_2 has a repeated color. Then we have u_4, u_5, u_n, u_{n-1} must be non-repeated color. We remove the vertices $\{u_1, u_2, u_3, u_n\}$ and add an edge u_4u_{n-1} in C_n. Therefore,we have the coloring $C|\langle[4, n-1]\rangle$ is a td-coloring with colors $\chi_{td}(C_n) - 2$. Therefore, $\chi_{td}(C_n) \geq 2 + \chi_{td}(C_{n-4}) \geq 2 + \chi_{td}(P_{n-4}) = \chi_{td}(P_n)$.

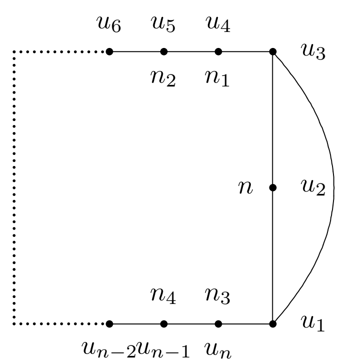


Fig.2

If x is a non-repeated color, then by Lemma 2.3, $\chi_{td}(P_n) \leq \chi_{td}(C_n)$. If there is no color class of the form $N(x)$, then $\chi_{td}(P_n) \leq \chi_{td}(C_n)$. □

Theorem 2.6 $\chi_{td}(C_n) = \chi_{td}(P_n)$, *for all good integers n.*

Proof The result follows from Corollary 2.2 and Lemmas 2.4 and 2.5. □

Remark Thus the $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 8, 12, 13, 15, 16, 17$ and $\forall n \geq 19$. It can be verified that $\chi_{td}(C_n) = \chi_{td}(P_n)$ for $n = 5, 6, 7, 9, 10, 14$ and that $\chi_{td}(C_n) = \chi_{td}(P_n) + 1$ for $n = 3, 11, 18$ and that $\chi_{td}(P_4) = \chi_{td}(C_4) + 1$.

References

[1] M.I.Jinnah and A.Vijayalekshmi, *Total Dominator Colorings in Graphs*, Ph.D Thesis 2010, University of Kerala, India.

[2] A.Vijayalekshmi, Total dominator colorings in Paths, *International Journal of Mathematical Combinatorics*, Vol 2 (2012), p. 89–95.

[3] F.Harary, *Graph Theory*, Addition - Wesley Reading Mass, 1969.

[4] Terasa W.Haynes, Stephen T.Hedetniemi, Peter J.Slater, *Domination in Graphs*, Marcel Dekker , New York, 1998.

[5] Terasa W.Haynes, Stephen T.Hedetniemi, Peter J. Slater, *Domination in Graphs - Advanced Topics*, Marcel Dekker,New York, 1998.

Math.Combin.Book Ser. Vol.4(2012), 97-102

p^*-Graceful Graphs

Teena Liza John and Mathew Varkey T.K.

(Department of Mathematics, T.K.M College of Engineering, Kollam-5, Kerala, India)

E-mail: teenavinu@gmail.com, mathewvarkeytk@gmail.com

Abstract: A labeling or numbering of a graph is an assignment of labels to the vertices of G that induces a number to each edge. In this paper we define p^*-graceful graphs and investigate some graphs based on this definition.

Key Words: Pentagonal numbers, p^*-graceful graphs, Comb graph, Twig graph, Banana trees.

AMS(2010): 05C78

§1. Introduction

Throughout this paper, by a graph we mean a simple finite graph without isolated vertices. For all the terminology and notations in Graph Theory, we follow [2] and for all terminology regarding labeling we follow [4].

Graceful labeling has been suggested by Bermond in [1]. A graph $G = (V, E)$ is numbered if each vertex v is assigned a non-negative integer $f(v)$ and each edge uv is attributed the absolute value of the difference of numbers of its end points, that is, $|f(u) - f(v)|$. The numbering is called graceful if further more, we have the following three conditions: (1) all the vertices are labeled with distinct integers; (2) the largest value of the vertex labels is equal to the number of edges, i.e $f(v) \in \{0, 1, \cdots, q\}$ for all $v \in V(G)$ and (3) the edges of G are distinctly labeled with integers from 1 to q.

In this paper we suggest a labeling called p^*-graceful labeling which is an analogue to graceful labeling and investigate the p^*-graceful nature of some graphs.

§2. P^* Graceful Labeling Graphs

Definition 2.1 *A labeling f of a graph G is one-one mapping from the vertex set of G into the set of integers. Let G be a graph with q edges. Let $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ be an injective function. Define the function $f_p^* : E(G) \to \{\omega^p(1), \omega^p(2), \cdots, \omega^p(q)\}$ such that $f_p^*(u, v) = |f_p(u) - f_p(v)|$. So f_p is said to be pentagonal graceful labeling of G and G is called a $p^* -$ graceful graph. Here $\omega^p(q) = \dfrac{q(3q-1)}{2}$ is the q^{th} pentagonal number.*

[1]Received August 28, 2012. Accepted December 16, 2012.

Theorem 2.1 *All paths are p^* graceful graphs.*

Proof Let $\{v_1, v_2, \cdots, v_n\}$ be the vertices of P_n, the path on n vertices. Define $f_p : V(P_n) \to \{0, 1, \cdots, \omega^p(n-1)\}$ such that

$$f_p(v_1) = 0;$$
$$f_p(v_{2i}) = f_p(v_{2i-1}) + \omega^p(q - (2i-2)) \quad \text{where } i = 1, 2, \cdots, \left\lfloor \frac{q}{2} \right\rfloor;$$
$$f_p(v_{2i+1}) = f_p(v_{2i}) - \omega^p(q - (2i-1)) \quad \text{where } i = 1, 2, \cdots, \left\lfloor \frac{q}{2} \right\rfloor.$$

Obviously, f_p is nothing but a pentagonal graceful labeling on P_n with $f_p^*(P_n) = \{\omega^p(1), \omega^p(2), \cdots, \omega^p(n-1)\}$. □

Theorem 2.2 *The graph nK_2 is p^*-graceful.*

Proof Let each K_2 be labeled with u_i, v_i where $1 \leqslant i \leqslant n$. Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ such that

$$f_p(u_i) = i - 1 \quad \text{if } 1 \leqslant i \leqslant n;$$
$$f_p(v_i) = w^p(n - (i-1)) + f_p(u_i) \quad \text{if } 1 \leqslant i \leqslant n.$$

Then $f_p(u_i) \neq f_p(u_j)$ for $i \neq j$. Otherwise, if $f_p(u_i) = f_p(u_j)$ then $i - 1 = j - 1$. Thus $i = j$, a contradiction.

Again, if $i \neq j, f_p(v_i) \neq f_p(v_j)$. Otherwise if $f_p(v_i) = f_p(v_j)$ then $\omega^p(q-(i-1)) + f_p(u_i) = \omega^p(q - (j-1)) + f_p(u_j)$, i.e, $\omega^p(q - (i-1)) - \omega^p(q - (j-1)) = f_p(u_j) - f_p(u_i) \neq 0$. Thus $\omega^p(q - (i-1)) \neq \omega^p(q - (j-1))$. Consequently, $f_p(v_i) \neq f_p(v_j)$. Hence f_p is one-one.

Also

$$\begin{aligned} |f_p(u_i) - f_p(v_i)| &= |f_p(u_i) - \omega^p(q - (i-1)) - f_p(u_i)| \\ &= \omega^p(q - (i-1)) \text{ for } i = 1, 2, \cdots, n. \end{aligned}$$

We will have $\omega^p(n), \omega^p(n-1), \cdots, \omega^p(1)$ as the edge labels. Hence the result. □

Definition 2.2 *Let T be a tree. Denote the tree obtained from T by considering two copies of T and adding an edge by $T_{(2)}$ and in general the graph obtained from $T_{(n-1)}$ and T by adding an edge between them is denoted by $T_{(n)}$. Now $T_{(1)}$ is just T.*

Corollary 2.1 *$T_{(n)}$ is a p^*-graceful graph.*

Let P_{2n+1} be a path on $2n+1$ vertices. Take $2m + 1 = \alpha$ is isomorphic copies of P_{2n+1}. Let w be a vertex which is adjacent to one end vertex of each copy. The newly obtained graph is a star with $2m+1$ spokes in which each spoke is a path of length $2n+1$ and is denoted by $S_{2n+1,2m+1}$. The degree of w is $2n+1$ and all the other vertices are of degree either 2 or 1. So this is a trivalent tree. In [5] Mathew Varkey proved that $S_{2k+1,2m+1}$ is a prime graph for all k and m. Now we prove the following.

Theorem 2.3 *The star $S_{2n+1,2m+1}$ is p^*-graceful for all $n, m \geqslant 1$.*

Proof Let P_{2n+1} be a path of length $2n$. Consider $\alpha = 2m+1$ isomorphic copies of P_{2n+1}. Adjoin a new vertex w to one end vertex of each copy of P_{2n+1}. Let $v_{i1} : i = 1, 2, \cdots, 2m+1$ be the vertices in the first level and $v_{ij} : i = 1, 2, \cdots, 2m+1$ and $j = 2, 3, \cdots, 2n+1$ be the remaining vertices of $S_{2n+1,2m+1}$. Define a function f_p from the vertex set of $S_{2n+1,2m+1}$ to the set of all non-negative integers less than or equal to number of edges of $S_{2n+1,2m+1}$ such that

$f_p(w) = 0, f_p(v_{i1}) = \omega^p(q - i + 1);\ i = 1, 2, \cdots, 2m+1$ and

$f_p(v_{ij}) = f_p(v_{i,j-1}) + (-1)^{j-1}\omega^p(q - (2m+1)(j-1) - (i-1))$ for $i = 1, 2, \cdots, 2m+1; j = 2, \cdots, 2n$.

Clearly f is injective. Hence $S_{2n+1,2m+1}$ is p^*-graceful. □

Corollary 2.2 *The star $S_{2n+1,2n+1}$ is p^*-graceful for all n.*

Corollary 2.3 *The star $S_{2n,2n}$ is p^*-graceful for all $n \geqslant 1$.*

Definition 2.3 *A caterpillar is a tree with the property that the removal of its end points leaves a path.*

Theorem 2.4 *A caterpillar $S(n_1, n_2, \cdots, n_m)$ is p^*-graceful.*

Definition 2.4 *The eccentricity $e(v)$ of a vertex v in a tree T is defined as $\max\{d(v,u) : u \in V(T)\}$ and the radius of T is the minimum eccentricity of the vertices.*

Definition 2.5 *A centre of a tree is a vertex of minimum eccentricity.*

Definition 2.6 *The neighborhood of vertex u is the set $N(u)$ consisting of all the vertices v which are adjacent with u. The closed neighborhood is defined as $N[u]$ and is given by $N[u] = N(u) \cup \{u\}$.*

A result by Jordan states that every tree has centre consisting of one point or two adjacent points. In this section we consider trees with exactly one centre.

Let $\{\alpha_1 K_{1,n_1}; \alpha_2 K_{1,n_2}; \cdots \alpha_p K_{1,n_p}\}$ be a family of stars where $\alpha_i K_{1,n_i}$ denotes α_i disjoint isomorphic copies of K_{1,n_i} for $i = 1, 2, \cdots, p$ and $\alpha_i \geqslant 1$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k = 1, 2, \cdots, n_i$ be the central and end vertices respectively of H_{ij}. Let w be a new vertex adjacent to u_{ij} for $j = 1, 2, \cdots, \alpha_i; i = 1, 2, \cdots, p$. We thus obtain a new tree of radius 2 with unique centre which we shall denote by $H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$. [see 5]. Now we prove the following theorem.

Theorem 2.5 *$H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$ is p^*-graceful.*

Proof Consider the family of stars $\alpha_i K_{1,n_i}$ for $i = 1, 2, \cdots, p$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k = 1, 2, \cdots, n_i$ be the central and end vertices respectively of H_{ij}. Let w be a new vertex adjacent to u_{ij} for $i = 1, 2, \cdots, p; j = 1, 2, \cdots, \alpha_i$ of each star.

Consider the mapping $f_p : V \to \{0, 1, \cdots, \omega^p(q)\}$ (where V is the vertex set and q is the

number of edges of $H_w^{(\alpha_1+\alpha_2+\cdots+\alpha_p)}$) defined as $f_p(w)=1, f_p(u_{11})=0$

$$\begin{aligned} f_p(u_{ij}) &= 0 \quad \text{for} \quad i=j=1 \\ &= \omega^p(j)+1 \quad \text{for} \quad i=1 \text{ and } j=2,3,\cdots,\alpha_1 \\ &= \omega^p(\alpha_1+\alpha_2+\cdots+\alpha_{i-1}+j)+1 \quad \text{for} \quad i=2,3,\ldots,p; j=1,2,\cdots\alpha_i \end{aligned}$$

$$\begin{aligned} f_p(v_{ijk}) &= \omega^p(q-(k-1)) \quad \text{for} \quad i=j=1; k=1,2,\cdots,n_1 \\ &= \omega^p(q-(j-1)n_1-(k-1))+f_p(u_{1j}) \text{ for } i=1; j=2,3,\cdots,\alpha_1; k=1,2,\cdots,n_1 \\ &= \omega^p\left(q-\sum_{l=1}^{i-1}\alpha_l n_l-(j-1)n_i-(k-1)\right)+f_p(u_{ij}) \\ &\quad \text{for } i=2,3,\cdots,p; j=1,2,\cdots,\alpha_p; k=1,2,\cdots,n_p \end{aligned}$$

Thus f_p is a one-one mapping which induces the edge labels $\{\omega^p(1),\omega^p(2),\ldots,\omega^p(q)\}$. Hence f_p is p^*- graceful labeling. Hence the theorem. □

Consider the family of stars $\{\alpha_1K_{1,n_1};\alpha_2K_{1,n_2};\cdots\alpha_pK_{1,n_p}\}$ where α_iK_{1,n_i} denotes α_i disjoint isomorphic copies of K_{1,n_i} for $i=1,2,\cdots,p$ and $\alpha_i\geqslant 1$. Let H_{ij} be the j^{th} isomorphic copy of K_{1,n_i} and u_{ij} and v_{ijk} for $k=1,2\cdots,n_i$ be the central and end vertices respectively of H_{ij}. Adjoin a new vertex w to one end vertex of each star. The tree thus obtained is a tree with unique centre and radius 3 and is denoted by $H_w^{*(\alpha_1+\alpha_2+\cdots+\alpha_p)}$. Trees of this kind are referred to as banana trees, by some authors.

Corollary 2.4 *$H_w^{*(\alpha_1+\alpha_2+\cdots+\alpha_p)}$ is p^*-graceful.*

Definition 2.7 *The comb graph is a graph obtained from a path P_n by attaching a pendant vertex to each vertex of P_n.*

Theorem 2.6 *The comb graph $G=P_n\Theta K_1$ is p^*-graceful.*

Proof Let $u_1,u_2,\cdots,u_n$ be the vertices of P_n and $v_1,v_2,\cdots,v_n$ be the pendant vertices attached to $u_i: i=1,2,\cdots,n$. Then the graph $G=P_n\Theta K_1$ has $2n$ vertices and $q=2n-1$ edges.

Define $f_p: V(G)\to\{0,1,\cdots,\omega^p(2n-1)\}$ such that

$$\begin{aligned} f_p(u_i) &= \omega^p(q-(i-1)+f_p(u_{i-1}) && \text{when } i \text{ is even} \\ &= f_p(u_{i-1})-\omega^p(q-(i-1)) && \text{when } i\neq 1 \text{ is odd} \\ &= 0 && \text{when } i=1 \end{aligned}$$

$$\begin{aligned} f_p(v_i) &= f_p(u_i)-\omega^p(q-(n+i-2)) && \text{when } i \text{ is even} \\ &= f_p(u_i)+\omega^p(q-(n+i-2)) && \text{when } i\neq 1 \text{ is odd} \\ &= \omega^p(q) && \text{when } i=1 \end{aligned}$$

Thus $f_p^*=\{\omega^p(1),\cdots,\omega^p(q)\}$. Hence G is p^*-graceful. □

Definition 2.8 *A twig is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertex of the path.*

Theorem 2.7 *The twig graphs are p^*-graceful.*

Proof Let $v_1, v_2, \cdots, v_n$ be the n vertices of P_n and $v_{ij}; i = 2, 3 \cdots, n-1;\ j = 1, 2$ be the pendant vertices attached to each v_i. Then the graph has $q = 3n - 5$ edges.

Define $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$
such that $f_p(v_1) = 0$

$$f_p(v_{2i}) = f_p(v_{2i-1}) + \omega^p(q - (2i-2)) \qquad i = 1, 2, \cdots, \left\lfloor \frac{n}{2} \right\rfloor;$$
$$f_p(v_{2i+1}) = f_p(v_{2i}) - \omega^p(q - (2i-1)) \qquad i = 1, 2, \cdots, \left\lfloor \frac{n-1}{2} \right\rfloor;$$
$$f_p(v_{ij}) = f_p(v_i) + (-1)^{i-1}\omega^p(q - (n-1) - 2(i-2) - (j-1)) \text{ for } i = 2, \cdots, n-1; j = 1, 2.$$

Hence $f_p^* = \{\omega^p(1), \cdots, \omega^p(q)\}$. Therefore G is p^*-graceful. □

Definition 2.9 *The graph $C_n \hat{\circ} K_{1,n}$ is obtained from C_n and $K_{1,n}$ by identifying any vertex of C_n with the central vertex of $K_{1,n}$.*

Theorem 2.8 *$C_3 \hat{\circ} K_{1,n}$ is p^*-graceful for $n \geqslant 5$.*

Proof Let $C_3 \hat{\circ} K_{1,n} = G$ and let v_1, v_2, v_3 be the vertices of C_3, $u_1, u_2, \cdots, u_n$ be the vertices of $K_{1,n}$. Let v_1 be the vertex to which $K_{1,n}$ is attached with. The mapping $f_p : V(G) \to \{0, 1, \cdots, \omega^p(q)\}$ where $q = n + 3$, defined by

$$\begin{aligned} f_p(u_i) &= \omega^p(i) \text{ for } i = 1, 2, 3 \\ &= \omega^p(i+1) \text{ for } i = 4, 5 \\ &= \omega^p(i+3) \text{ for } i = 6, 7, \cdots, q \end{aligned}$$

and $f_p(v_1) = 0, f_p(v_2) = 22 = \omega^p(4), f_p(v_3) = 92 = \omega^p(8)$ is p^*-graceful.

Further, the theorem is true only for $n \geqslant 5$. Since $C_3 \hat{\circ} K_{1,n}$ to have a p^*-graceful labeling we should have the pentagonal numbers $\omega^p(4)$ and $\omega^p(8)$ for the vertices of C_3 which is possible only with $n \geqslant 5$. □

§3. Graphs That Are Not p^*-Graceful.

Theorem 3.1 *Wheels are not p^*-graceful.*

Proof As the central vertex of a wheel is attached to all other vertices, 0 cannot be assigned to it, for if, 0 is the central label then the attaching vertices , that is, all the remaining vertices should have pentagonal numbers as their respective labels which in turn leads to non-pentagonal numbers as edge labels, contradicting the definition. Again if we assign 0 to any other vertex, then we will have to label its adjacent three vertices with pentagonal numbers which again generates non-pentagonal numbers as edge labels. Thus in no way wheels are p^*-graceful. □

Definition 3.1 *The Helm H_n is the graph obtained from a wheel by attaching a pendant vertex at each vertex of the n-cycle.*

Corollary 3.1 *The Helm H_n is not p^*-graceful.*

Definition 3.2 *The Fan graph $P_n + K_1$ is a graph obtained by joining the path P_n with the complete graph K_1.*

Corollary 3.2 *The Fan graph $P_n + K_1$ is not p^*-graceful.*

Definition 3.3 *Let W_n be a wheel with $n+1$ vertices. Attach a pendant edge to each rim vertex of W_n. Join each pendant vertex with the central vertex of the wheel. This graph is called the Flower graph denoted by F_n.*

Corollary 3.3 *The Flower graph F_n is not p^*-graceful.*

Remark Further research on the topic is pursued.

References

[1] Bermond J.C, Graceful graphs, Radio antennae and French windmills, *Graph Theory and Combinatorics*, Pitman, London (1979) 13-37.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading M.A. 1969.

[3] Golomb S.W, How to number a graph, *Graph Theory and Computing*, R.C.Read,ed., Academic Press 1972,23-37.

[4] Joseph A. Gallian, A dynamic survey of Graph Labeling, *The Electronic Journal of Combinatorics*, 18(2011), 1-175.

[5] Mathew Varkey T.K, *Some Graph Theoretic Operations associated with Graph Labelings*, Ph.D Thesis (2000), University of Kerala.

[6] Rosa A, On certain valuations of the vertices of a graph, *Theory of Graphs* (International Symposium, Rome, July 1966) Gordon and Breach N.Y and Dunod Paris (1967), 349-355.

Math.Combin.Book Ser. Vol.4(2012), 103-115

Magic Graphoidal on Join of Two Graphs

A.Nellai Murugan

(Department of Mathematics, V.O.Chidambaram College, Tuticorin - 628008, Tamil Nadu, India)

E-mail: anellai.vocc@gmail.com

Abstract: B. D. Acharya and E. Sampathkumar [1] defined graphoidal cover as partition of edge set of G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G. Let $G = \{V, E\}$ be a graph and let ψ be a graphoidal cover of G. Define $f : V \cup E \to \{1, 2, \ldots, p + q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in ψ with

$$f^*(P) = f(v_0) + f(v_n) + \sum_{1}^{n} f(v_{i-1}v_i) = k,$$

a constant, where f^* is the induced labeling on ψ. Then, we say that G admits ψ - magic graphoidal total labeling of G. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ- magic graphoidal total labeling. In this paper, we proved that Wheel $W_n = C_{n-1} + K_1, K_2 + mK_1, K_2' + mk_1$, Fan $P_n + K_1$, Double Fan $P_n + 2K_1$ and Parachute $W_{n,2} = P_{2,n-2}$ are magic graphoidal.

Key Words: Graphoidal cover, magic graphoidal, graphoidal constant.

AMS(2010): 05C78

§1. Introduction

By a graph we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted are by $V(G)$ and $E(G)$ respectively. Wheel $W_n = C_{n-1} + K_1$ is a wheel, $K_2 + mK_1$ is a graph obtained by joining m isolated vertices to each end of K_2, a graph of path of length $1, \overline{K}_2 + mK_1$ is a graph obtained by joining m isolated vertices to each end of $\overline{K}_2, P_n + K_1$ is a fan, $P_n + 2K_1$ is a double fan and $W_{n,2} = P_{2,n-2}$ is Parachute. Terms and notations not used here are as in [3].

§2. Preliminaries

Let $G = \{V, E\}$ be a graph with p vertices and q edges. A graphoidal cover ψ of G is a collection of (open) paths such that

(i) every edge is in exactly one path of ψ

(ii) every vertex is an interval vertex of almost one path in ψ.

[1]Received September 25, 2012. Accepted December 18, 2012.

We define $\gamma(G) = \min_{\psi \in \zeta} |\psi|$, where ζ is the collection of graphoidal covers ψ of G and γ is graphoidal covering number of G.

Let ψ be a graphoidal cover of G. Then we say that G admits ψ-magic graphoidal total labeling of G if there exists a bijection $f : V \cup E \to \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0 v_1 v_2 \ldots v_n)$ in ψ, then $f^*(P) = f(v_0) + f(v_n) + \sum_1^n f(v_{i-1}v_i) = k$, a constant, where f^* is the induced labeling of ψ. A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ-magic graphoidal total labeling. In this paper, we proved that Wheel $W_n = C_{n-1} + K_1, K_2 + mK_1, K_2' + mk_1$, Fan $P_n + K_1$, Double Fan $P_n + 2K_1$ and Parachute $W_{n,2} = P_{2,n-2}$ are magic graphoidal.

Result 2.1([11]) *Let $G = (p, q)$ be a simple graph. If every vertex of G is an internal vertex in ψ, then $\gamma(G) = q - p$.*

Result 2.2([11]) *If every vertex v of a simple graph G, where degree is more than one, i.e., $d(v) > 1$, is an internal vertex of ψ is minimum graphoidal cover of G and $\gamma(G) = q - p + n$, where n is the number of vertices having degree one.*

Result 2.3([11]) *Let G be (p, q) a simple graph. Then $\gamma(G) = q - p + t$, where t is the number of vertices which are not internal.*

Result 2.4([11]) *For any k-regular graph $G, k \geq 3, \gamma(G) = q - p$.*

Result 2.5([11]) *For any graph G with $\delta \geq 3, \gamma(G) = q - p$.*

§3. Magic Graphoidal on Join of Two Graphs

Theorem 3.1 *The graph $K_2 + mK_1$ is magic graphoidal.*

Proof Let $G = K_2 + mK_1$ with $V(G) = \{u, v, u_i, 1 \leq i \leq m\}$ and $E(G) = \{(uv)\} \cup \{(uu_i)(vu_i), 1 \leq i \leq m\}$. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u) = 2m+2; \quad f(v) = 2m+3; \quad f(uv) = 2m+1,$$
$$f(uu_i) = i, \ 1 \leq i \leq m,$$
$$f(vu_i) = 2m+1-i, \ 1 \leq i \leq m.$$

Let $\psi = \{(uv), [(uu_iv) : 1 \leq i \leq m]\}$. Then

$$\begin{aligned} f^*[(uv)] &= f(u) + f(v) + f(uv) \\ &= 2m+2+2m+3+2m+1 \\ &= 6m+6. \qquad (1) \end{aligned}$$

For integers $1 \le i \le m$,

$$\begin{aligned} f^*[(uu_iv)] &= f(u) + f(v) + f(uu_i) + f(u_iv) \\ &= 2m + 2 + 2m + 3 + i + 2m + 1 - i \\ &= 6m + 6. \end{aligned} \tag{2}$$

From (1) and (2), we conclude that ψ is minimum magic graphoidal cover. Hence, K_2+mK_1 is magic graphoidal. □

For example, the magic graphoidal cover of $K_2 + 7K_1$ is shown in Figure 1,

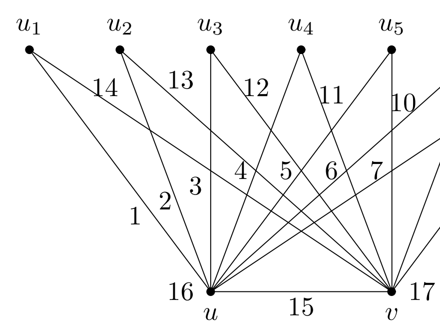


Figure 1 $K_2 + 7K_1$

where, $\psi = \{(uv), (uu_1v), (uu_2v), (uu_3v), (uu_4v), (uu_5v), (uu_6v), (uu_7v)\}, \gamma = 8$ and $K = 48$.

Theorem 3.2 *Parachute $W_{n,2} = P_{2,n-2}$ is magic graphoidal.*

Proof Let $G = W_{n,2}$ with $V(G) = \{u_i : 1 \le i \le n], v\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \le i \le n-1\} \cup \{(u_1u_n)\} \cup \{(vu_i) : 1 \le i \le 2\}$.

Case 1 n is odd.

Let $\psi = \left\{ \left(u_nvu_1u_2 \ldots u_{\frac{n+1}{2}} \right), \left(u_1u_nu_{n-1}u_{n-2} \ldots u_{\frac{n+1}{2}} \right) \right\}$.

Subcase 1.1 $n = 1(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = 4, \quad f(u_n) = 1, \quad f(u_1u_n) = 5,$$

$$f(vu_1) = 6, \quad f(vu_n) = 2, \quad f\left(u_{\frac{n+1}{2}}\right) = p + q$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+5 & \text{if } i = 1(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \\ 2i+6 & \text{if } i = 0(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \end{cases}$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \le i \le \dfrac{n-1}{2} \end{cases}$$

Then

$$
\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \ldots u_{\frac{n+1}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n+1}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}\right) \\
&= 1 + p + q + 2 + 6 + \sum_{i=1,3}^{\frac{n+1}{2}-2} (2i+5) + \sum_{i=2,4}^{\frac{n+1}{2}-1} (2i+6) \\
&= p + q + 9 + 11\left(\frac{n-1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \ldots u_{\frac{n+1}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n+1}{2}}\right) + f(u_1 u_n) + f(u_n u_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n-1}{2}} u_{\frac{n+1}{2}}\right) \\
&= 4 + p + q + 5 + \sum_{i=1,3}^{\frac{n+1}{2}-2} (2i+6) + \sum_{i=2,4}^{\frac{n+1}{2}-1} (2i+5) \\
&= p + q + 9 + 11\left(\frac{n-1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \qquad (4)
\end{aligned}
$$

From (3) and (4), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{5,2}$ is shown in Figure 2.

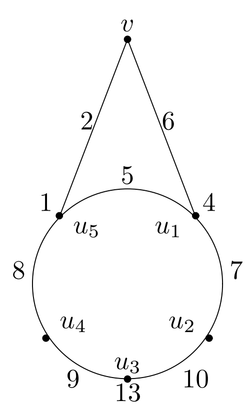


Figure 2 $W_{5.2}$

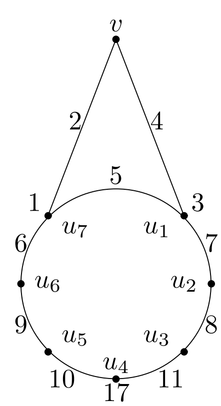


Figure 3 $W_{7.2}$

Subcase 1.2 $n = 3(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$
\begin{aligned}
&f(u_1) = 3; \quad f(u_n) = 1; \quad f(vu_n) = 2, \\
&f(vu_1) = 4, \quad f(u_1u_n) = 5; \quad f\left(u_{\frac{n+1}{2}}\right) = p + q,
\end{aligned}
$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+5 & \text{if } i = 1(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \\ 2i+4 & \text{if } i = 0(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \end{cases}$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \leq i \leq \dfrac{n-1}{2} \end{cases}$$

Then

$$\begin{aligned} f^*\left[\left(u_nvu_1u_2\ldots u_{\frac{n+1}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n+1}{2}}\right) + f(u_nv) \\ &\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n-1}{2}}u_{\frac{n+1}{2}}\right) \\ &= 1 + p + q + 2 + 4 + \sum_{i=1,3}^{\frac{n+1}{2}-1}(2i+5) + \sum_{i=2,4}^{\frac{n+1}{2}-2}(2i+4) \\ &= p + q + 3 + 9\left(\frac{n+1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \end{aligned} \tag{5}$$

$$\begin{aligned} f^*\left[\left(u_1u_nu_{n-1}u_{n-2}\ldots u_{\frac{n+1}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n+1}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\ &\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n-1}{2}}u_{\frac{n+1}{2}}\right) \\ &= 3 + p + q + 5 + \sum_{i=1,3}^{\frac{n+1}{2}-1}(2i+4) + \sum_{i=2,4}^{\frac{n+1}{2}-2}(2i+5) \\ &= p + q + 3 + 9\left(\frac{n+1}{4}\right) + \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \end{aligned} \tag{6}$$

From (5) and (6), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{7,2}$ is shown in Figure 3.

Case 2 n is even.

Let $\psi = \left\{\left(u_nvu_1u_2\ldots u_{\frac{n}{2}}\right), \left(u_1u_nu_{n-1}u_{n-2}\ldots u_{\frac{n}{2}}\right)\right\}$.

Subcase 2.1 $n = 2(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = 2; \quad f(vu_1) = 7; \quad f(u_n) = 1,$$
$$f(u_1u_n) = 4; \quad f(vu_n) = 3; \quad f(u_nu_{n-1}) = 5,$$
$$f\left(u_{\frac{n}{2}}\right) = p + q,$$

$$f(u_iu_{i+1}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \leq i < \dfrac{n}{2} \\ 2i+7 & \text{if } i = 0(mod\ 2), 1 \leq i < \dfrac{n}{2} \end{cases},$$

$$f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 3 \leq i \leq \dfrac{n}{2} \\ 2i+5 & \text{if } i = 0(mod\ 2), 2 \leq i \leq \dfrac{n}{2} \end{cases}.$$

Then,

$$
\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \dots u_{\frac{n}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n}{2}-1}u_{\frac{n}{2}}\right) \\
&= 1 + p + q + 3 + 7 + \sum_{i=1,3}^{\frac{n}{2}-2}(2i+6) + \sum_{i=2,4}^{\frac{n}{2}-1}(2i+7) \\
&= p + q + 9 + 11 + \frac{13}{2}\left(\frac{n}{2}-1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (7) \\
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \dots u_{\frac{n}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) \\
&= 2 + p + q + 4 + 5 + \sum_{i=2,4}^{\frac{n}{2}-1}(2i+5) + \sum_{i=1,3}^{\frac{n}{2}}(2i+4) \\
&= p + q + 11 + \frac{13}{2}\left(\frac{n}{2}-1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (8)
\end{aligned}
$$

From (7) and (8), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{6,2}$ is shown in Figure 4.

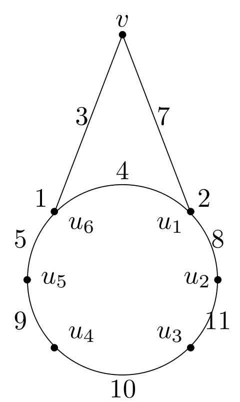


Figure 4 $W_{6.2}$

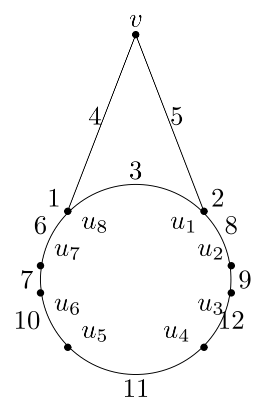


Figure 5 $W_{8.2}$

Subcase 2.2 $n = 0(mod\ 4)$.

Define $f : V \cup E \to \{1, 2, \dots, p+q\}$ by

$$
\begin{aligned}
&f(u_1) = 2; \quad f(u_n) = 1; f\left(u_{\frac{n}{2}}\right) = p + q, \\
&f(u_nu_{n-1}) = 6; \quad f(vu_1) = 5; \quad f(vu_n) = 4; \quad f(u_1u_n) = 3, \\
&f(u_iu_{i+1}) = \begin{cases} 2i+6 & \text{if } i = 1(mod\ 2), 1 \le i \le \dfrac{n}{2}-1 \\ 2i+5 & \text{if } i = 0(mod\ 2), 1 \le i \le \dfrac{n}{2}-1 \end{cases}, \\
&f(u_{n+1-i}u_{n-i}) = \begin{cases} 2i+4 & \text{if } i = 1(mod\ 2), 3 \le i \le \dfrac{n}{2}-1 \\ 2i+3 & \text{if } i = 0(mod\ 2), 2 \le i \le \dfrac{n}{2} \end{cases}.
\end{aligned}
$$

Then

$$
\begin{aligned}
f^*\left[\left(u_n v u_1 u_2 \dots u_{\frac{n}{2}}\right)\right] &= f(u_n) + f\left(u_{\frac{n}{2}}\right) + f(u_n v) \\
&\quad + f(vu_1) + f(u_1u_2) + \cdots + f\left(u_{\frac{n}{2}-1}u_{\frac{n}{2}}\right) \\
&= 1 + p + q + 4 + 7 + \sum_{i=1,3}^{\frac{n}{2}-1}(2i+6) + \sum_{i=2,4}^{\frac{n}{2}-2}(2i+5) \\
&= p + q + 5 + 11\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (9) \\
f^*\left[\left(u_1 u_n u_{n-1} u_{n-2} \dots u_{\frac{n}{2}}\right)\right] &= f(u_1) + f\left(u_{\frac{n}{2}}\right) + f(u_1u_n) + f(u_nu_{n-1}) \\
&\quad + f(u_{n-1}u_{n-2}) + \cdots + f\left(u_{\frac{n}{2}+1}u_{\frac{n}{2}}\right) \\
&= 2 + p + q + 3 + 6 + \sum_{i=2,4}^{\frac{n}{2}}(2i+3) + \sum_{i=1,3}^{\frac{n}{2}-1}(2i+4) \\
&= p + q + 5 + 11\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right) \qquad (10)
\end{aligned}
$$

From (9) and (10), we conclude that ψ is minimum magic graphoidal cover. Hence, $W_{n,2}$ is magic graphoidal. For example, the magic graphoidal cover of $W_{8,2}$ is shown in Figure 5. Hence, Parachute admits magic graphoidal. □

Theorem 3.3 *A Fan $P_n + K_1$ is magic graphoidal for $n \equiv 0(mod2)$.*

Proof If $n = 2$, a fan becomes K_3. If $n = 4$, a labeling on $P_4 + K_1$ is shown in Figure 6,

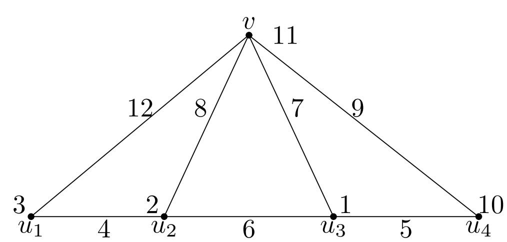


Figure 6 $P_4 + K_1$

where, $\psi = \{(u_1v), (u_1u_2v), (u_2u_3v), (u_3u_4v)\}, \gamma = 4$ and $K = 26$.

If $n > 4$, let $G = P_n + K_1$ with $V(G) = \{v, u_i : 1 \le i \le n\}$ and $E(G) = \{(vu_i) : 1 \le i \le n\} \cup \{(u_iu_{i+1}) : 1 \le i \le n-1\}$ Let $\psi = \{(u_1v), (u_iu_{i+1}v) : 1 \le i \le n-1\}$. Define $f : V \cup E \to \{1, 2, \dots, p+q\}$ by

$$f(v) = p + q - 1, \quad f(u_n) = p + q - 2, \quad f(u_1) = 3\left(\frac{n-6}{2}\right) + 6,$$

$$f(u_i) = f(u_1) + (i-1),\ i = 2, 3, \quad f(vu_1) = p + q, \quad f(u_iu_{i+1}) = 7 - 2i, \quad 1 \le i \le 3$$

$$f(u_{n-i}) = f(u_1) + 2 + i,\ 1 \le i \le \frac{n}{2} - 1, \quad f(u_{3+i}) = f(u_1) - i \text{ if } n > 6, 1 \le i \le \frac{n}{2} - 3,$$

$$f(u_{n+1-i}u_{n-i}) = n - 2i,\ 1 \le i \le \frac{n}{2} - 1, \quad f(u_{3+i}u_{4+i}) = 5 + 2i,\ 1 \le i \le \frac{n}{2} - 3,$$

$$f(vu_{i+1}) = p+q-6+i,\ 1 \le i \le 3, \quad f(vu_{n+1-i}) = f(u_1) + \frac{n+2}{2} + i,\ 1 \le i \le n-4.$$

Then

$$\begin{aligned} f*[(u_1v)] &= f(u_1)+f(v)+f(u_1v) \\ &= \frac{3}{2}(n-6)+6+3n-1+3n = 7n+\frac{n}{2}-4. \end{aligned} \tag{11}$$

For $2 \le i \le 3$,

$$\begin{aligned} f^*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6+(i-1)+3n-1+7-2i+3n-6+i = 7n+\frac{n}{2}-4. \end{aligned} \tag{12}$$

For $4 \le i \le \frac{n}{2}-3$,

$$\begin{aligned} f^*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6-(i-3)+3n-1+5+2(i-3) \\ &\quad +3\left(\frac{n-6}{2}\right)+6+\frac{n+2}{2}+n-i = 7n+\frac{n}{2}-4. \end{aligned} \tag{13}$$

For $\frac{n}{2}-3 < i \le n-1$,

$$\begin{aligned} f*[(u_iu_{i+1}v)] &= f(u_i)+f(v)+f(u_iu_{i+1})+f(u_{i+1}v) \\ &= 3\left(\frac{n-6}{2}\right)+6+2+n-i+3n-1+n-2(n-i) \\ &\quad +3\left(\frac{n-6}{2}\right)+6+\frac{n+2}{2}+n-i = 7n+\frac{n}{2}-4. \end{aligned} \tag{14}$$

From (11), (12), (13) and (14), we conclude that ψ is minimum magic graphoidal cover. Hence, a Fan $P_n + K_1$, (n-even, $n > 4$) is magic graphoidal. □

For example, the magic graphoidal cover of $P_{10} + K_1$ is shown in Figure 7,

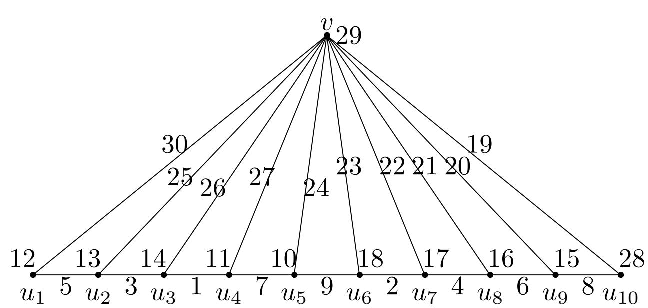


Figure 7 $P_{10} + K_1$

where, $\psi = \{(u_1v), (u_1u_2v), (u_2u_3v), (u_3u_4v), (u_4u_5v), (u_5u_6v), (u_6u_7v), (u_7u_8v), (u_8u_9v), (u_9u_{10}v)\}$, $\gamma = 10$ and $K = 71$.

Theorem 3.4 *A double Fan $P_n + 2K_1$is magic graphoidal.*

Proof Let $G = P_n + 2K_1$ with $V(G) = \{v, w\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \leq i \leq n-1\} \cup \{(vu_i) \cup (wu_i) : 1 \leq i \leq n\}$. The discussion is divided into to cases following.

Case 1 n is odd.

In this case, define $f : V \cup E \to \{1, 2, \ldots, 4n+1\}$ by

$$f(w) = 4n+1;\quad f(u_iv) = i,\ 1 \leq i \leq n;\quad f(u_iw) = 2n+1-i,\ 1 \leq i \leq n,$$
$$f(u_2i) = 2n+i,\ 1 \leq i \leq \frac{n-1}{2}; f(v) = \frac{5n+1}{2}$$
$$f(u_{2i-1}) = \frac{5n+1}{2} + i,\ 1 \leq i \leq \frac{n+1}{2};\quad f(u_{n+1-i}u_{n-i}) = 3n+1+i,\ 1 \leq i \leq n-1.$$

Let $\psi = \{(u_iu_{i+1}) : 1 \leq i \leq n-1\} \cup \{(vu_iw) : 1 \leq i \leq n\}$. Then, for $1 \leq i \leq n-1, i \equiv 1(mod\, 2)$,

$$\begin{aligned} f^*[(u_iu_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) \\ &= \frac{5n+1}{2} + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+5}{2}; \end{aligned} \tag{15}$$

for $1 \leq i \leq n-1, i \equiv 0(mod\ 2)$.

$$\begin{aligned} f^*[(u_iu_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_iu_{i+1}) \\ &= 2n + \frac{i}{2} + \frac{5n+1}{2} + \frac{i+2}{2} + 3n + 1 + n - i = \frac{17n+5}{2}; \end{aligned} \tag{16}$$

for $1 \leq i \leq n$,

$$\begin{aligned} f[(vu_iw)] &= f(v) + f(w) + f(vu_i) + f(u_iw) \\ &= \frac{5n+1}{2} + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+5}{2} \end{aligned} \tag{17}$$

From (15), (16) and (17), we conclude that ψ is minimum magic graphoidal cover. Hence, a double Fan P_n+2K_1 (n-odd) is magic graphoidal. For example, the magic graphoidal cover of P_5+2K_1 is shown in Figure 8 with $\psi = \{(vu_1w), (vu_2w), (vu_3w), (vu_4w), (vu_5w), (u_1u_2), (u_2u_3), (u_3u_4), (u_4u_5)\}, \gamma = 9$ and $K = 45$.

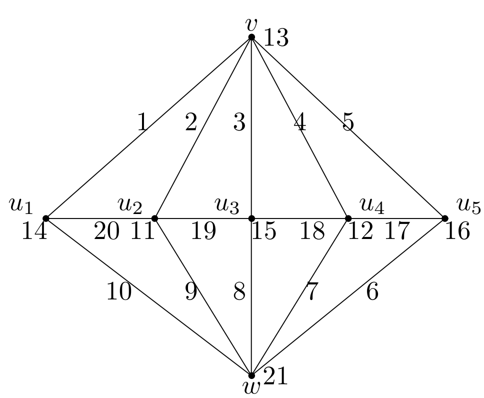


Figure 8 $P_5 + 2K_1$

Case 2 n is even.

In this case, define $f : V \cup E \to \{1, 2, \ldots, 4n+1\}$ by

$$f(w) = 4n+1, \quad f(u_i v) = i,\ 1 \le i \le n, \quad f(u_i w) = 2n+1-i,\ 1 \le i \le n$$
$$f(u_2 i) = 2n+i,\ 1 \le i \le \frac{n}{2}, \quad f(v) = \frac{5n}{2}+1$$
$$f(u_{2i-1}) = \frac{5n}{2}+1+i,\ 1 \le i \le \frac{n}{2}, \quad f(u_{n+1-i}u_{n-i}) = 3n+1+i,\ 1 \le i \le n-1.$$

Let $\psi = \{[(u_i u_{i+1}) : 1 \le i \le n-1], [(vu_i w) : 1 \le i \le n]\}$. Then, for $1 \le i \le n-1, i \equiv 1(mod\ 2)$,

$$\begin{aligned} f^*[(u_i u_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) \\ &= \frac{5n}{2} + 1 + \frac{i+1}{2} + 2n + \frac{i+1}{2} + 3n + 1 + n - i = \frac{17n+6}{2}; \end{aligned} \tag{18}$$

for $1 \le i \le n-1, i \equiv 0(mod\ 2)$

$$\begin{aligned} f^*[(u_i u_{i+1})] &= f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) \\ &= 2n + \frac{i}{2} + \frac{5n}{2} + 1 + \frac{i+2}{2} + 3n + 1 + n - i = \frac{17n+6}{2}; \end{aligned} \tag{19}$$

for $1 \le i \le n$,

$$\begin{aligned} f^*[(vu_i w)] &= f(v) + f(w) + f(vu_i) + f(u_i w) \\ &= \frac{5n}{2} + 1 + 4n + 1 + i + 2n + 1 - i = \frac{17n+6}{2} \end{aligned} \tag{20}$$

From (18), (19) and (20), we conclude that ψ is minimum magic graphoidal cover. Hence, a double Fan $P_n + 2K_1$ (n-even) is magic graphoidal. For example, the magic graphoidal cover of $P_6 + 2K_1$ is shown in Figure 9,

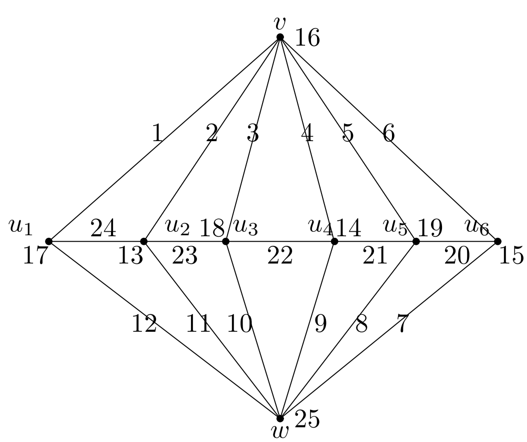


Figure 9 $P_6 + 2K_1$

where, $\psi = \{(vu_1w), (vu_2w), (vu_3w), (vu_4w), (vu_5w), (vu_6w), (u_1u_2), (u_2u_3), (u_3u_4), (u_4u_5), (u_5u_6)\}$, $\gamma = 11$ and $K = 54$. □

Theorem 3.5 *A wheel $W_n = C_{n-1} + K_1$ (n- even) is magic graphoidal.*

Proof Let $G = W_n$ with $V(G) = \{v, u_i : 1 \le i \le n-1\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \le i \le n-2\} \cup \{(u_1u_{n-1})\} \cup \{(vu_i) : 1 \le i \le n-1\}$. Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

$$f(u_1) = n-2;\quad f(u_{i+1}) = 2i-1,\ 1 \le i \le \frac{n}{2},$$
$$f\left(u_{\frac{n}{2}+1+i}\right) = 2i,\ 1 \le i \le \frac{n}{2}-2,\quad f(v) = p+q,$$
$$f(u_iu_{i+1}) = 3\frac{n}{2} - i,\ 1 \le i \le \frac{n}{2};\quad f(vu_i) = 3n-2-i,\ 1 \le i \le n-1,$$
$$f\left(u_{\frac{n}{2}+i}u_{\frac{n}{2}+1+i}\right) = 2n-1-i,\ 1 \le i \le \frac{n}{2}-1,\ \text{where } u_n = u_1.$$

Let $\psi = \{[(vu_iu_{i+1}) : 1 \le i \le n-2], (vu_{n-1}u_1)\}$. Then, for $1 \le i \le \frac{n}{2}$,

$$\begin{aligned} f^*[(vu_iu_{i+1})] &= f(v) + f(u_{i+1}) + f(vu_i) + f(u_iu_{i+1}) \\ &= 3n-2+2i-1+3n-2-i+\frac{3}{2}n - i = 7n + \frac{n}{2} - 5; \end{aligned} \tag{21}$$

for $\frac{n}{2} + 1 \le i \le n-2$,

$$\begin{aligned} f^*[(vu_iu_{i+1})] &= f(v) + f(u_{i+1}) + f(vu_i) + f(u_iu_{i+1}) \\ &= 3n-2+2\left(i-\frac{n}{2}\right)+3n-2-i+2n-1-\left(i-\frac{n}{2}\right) = 7n+\frac{n}{2}-5 \end{aligned} \tag{22}$$

$$\begin{aligned} f^*[(vu_{n-1}u_1)] &= f(v) + f(u_1) + f(vu_{n-1}) + f(u_{n-1}u_1) \\ &= 3n-2+n-2+3n-2-(n-1)+2n-1-\left(\frac{n}{2}-1\right) = 7n+\frac{n}{2}-5. \end{aligned} \tag{23}$$

From (21), (22) and (23), we conclude that ψ is minimum magic graphoidal cover. Hence, a wheel W_n (n-even) is magic graphoidal. □

For example, the magic graphoidal cover of W_8 is shown in Figure 10,

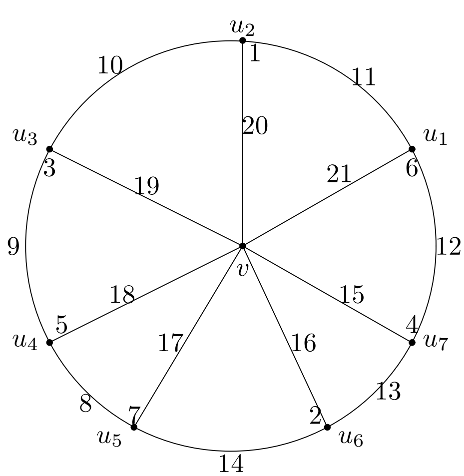


Figure 10 W_8

where, $\psi = (vu_1u_2), (vu_2u_3), (vu_3u_4), (vu_4u_5), (vu_5u_6), (vu_6u_7), (vu_7u_1), \gamma = 7$ and $K = 55$.

Theorem 3.6 *The graph $\overline{K}_2 + mK_1$ is magic graphoidal.*

Proof Let $G = \overline{K}_2 + mK_1$ with $V(G) = \{u_1, u_2, [v_i : 1 \leq i \leq m]\}$ and $E(G) = \{(u_1v_i) : 1 \leq i \leq m\} \cup \{(u_2v_i) : 1 \leq i \leq m\}$. Define $f : V \cup E \to \{1, 2, \ldots, 3m+2\}$ by

$$f(u_1) = 2m+1; \quad f(u_1v_i) = i, \ 1 \leq i \leq m,$$
$$f(u_2) = 2m+2; \quad f(u_2v_i) = 2m+1+i, \ 1 \leq i \leq m-1.$$

Let $\psi = \{(u_1v_iu_2) : 1 \leq i \leq m\}$. Then,

$$\begin{aligned} f^*[(u_1v_iu_2)] &= f(u_1) + f(u_2) + f(u_1v_i) + f(v_iu_2) \\ &= 2m+1+2m+2+i+2m+1-i = 6m+4. \end{aligned}$$

Thus, $f^*[(u_1v_iu_2)]$ is independent of i, depends only on m. So it is a constant. Therefore, $\overline{K}_2 + mK_1$ admits a ψ-magic total labeling. Hence, $\overline{K}_2 + mK_1$ is magic graphoidal. $\square$

For example, the magic graphoidal cover of $\overline{K}_2 + 6K_1$ is shown in Figure 11,

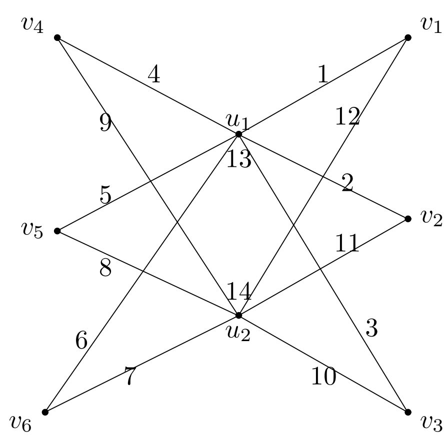


Figure 11 $\overline{K}_2 + 6K_1$

where, $\psi = \{(u_1v_1u_2), (u_1v_2u_2), (u_1v_3u_2), (u_1v_4u_2), (u_1v_5u_2), (u_1v_6u_2)\}, \gamma = 6, K = 40.$

References

[1] B.D.Acarya and E.Sampath Kumar, Graphoidal covers and Graphoidal covering number of a Graph, *Indian J.Pure Appl. Math.*, **18**(10) (1987), 882–890.

[2] J.A.Gallian, A dynamic survey of graph labeling, *The Electronic journal of Coimbinotorics*, **6** (2001) # DS6.

[3] F.Harary, *Graph Theory*, Addition - Wesley Publishing Company Inc, USA, 1969.

[4] A.Nellai Murugan and A.Nagarajan, *Magic graphoidal on trees*, (Communicated).

[5] A.Nellai Murugan and A.Nagarajan, *Magic graphoidal on cycle related graphs*, (Communicated).

[6] A.Nellai Murugan and A.Nagarajan, Magic graphoidal on path related graphs, *Advance Journal of Physical Sciences*, Vol.1(2), December 2012, pp 14-25.

[7] A.Nellai Murugan and A.Nagarajan, *On magic graphoidal graphs*, (Communicated).

[8] A.Nellai Murugan and A.Nagarajan, *Magic graphoidal on product graphs*, (Communicated).

[9] A.Nellai Murugan and A.Nagarajan, *Magic graphoidal on special types of graphs*, (Communicated).

[10] A.Nellai Murugan and A.Nagarajan, *Magic graphoidal on special class of graphs*, (Communicated).

[11] C.Packiam and S.Arumugam, On the Graphoidal covering number of a Graph, *Indian J.Pure Appl. Math.*, **20** (1989), 330–333.

Math.Combin.Book Ser. Vol.4(2012), 116-123

The Crossing Number of The Generalized Petersen Graph $P[3k-1,k]$

Zhidong Zhou

(Department of Mathematics and Computer Science, Hunan Normal University, Changsha, 410081, P.R.China)

Jing Wang

(Department of information and Computer Science, Changsha University, Changsha, 410003, P.R.China)

E-mail: zzdongwww@163.com

Abstract: The crossing number of a graph is the least number of crossings of edges among all drawings of the graph in the plane. In this paper, we investigate the crossing number of the generalized Petersen graph $P[3k-1,k]$ and get the result that $k \leq cr(P[3k-1,k]) \leq k+1$ for $k \geq 3$.

Key Words: crossing number, generalized Petersen graph, Cartesian product, Smarandache $\mathscr{P}$-drawing.

AMS(2010): 05C10, 05C62

§1. Introduction

Let $G=(V,E)$ be a simple graph with vertex set V and edge set E. A *Smarandache $\mathscr{P}$-drawing* of a graph G for a graphical property $\mathscr{P}$ is such a good drawing of G on the plane with minimal intersections for its each subgraph $H \in \mathscr{P}$, which is said to be *optimal* if $\mathscr{P}=G$ with minimized crossings. The crossing number $cr(G)$ of a simple graph G is defined as the minimum number of edge crossings in a drawing of G in the plane. A drawing with the minimum number of crossings(an optimal drawing) must be a good drawing; that is, each two edges have at most one point in common, which is either a common end-vertex or a crossing. Moreover, no three edges cross in a point. Let D be a good drawing of the graph G, we denote the number of crossings in D by $cr_D(G)$.

The generalized Petersen graph $P[m,n]$ is defined to be the graph of order $2m$ whose vertex set is $\{u_1,u_2,\cdots,u_m;x_1,x_2,\cdots,x_m\}$ and edge set is $\{u_iu_{i+1},u_ix_i,x_ix_{i+n}, i=1,2,\cdots,m;$ addition modulo m$\}$. The Cartesian product of two graph G_1 and G_2, denoted by $G_1\times G_2$, has the vertex set $V(G_1\times G_2)=V(G_1)\times V(G_2)$, edge set $E(G_1\times G_2)=\{(u_i,u_j)(u_h,u_k)|u_i=u_h$ and $v_jv_k\in E(G_2)$; or $v_j=v_k$ and $u_iu_h\in V(G_1)\}$. In a drawing D, if an edge is not crossed by any other edge, we say that it is clean in D; if it is crossed by at least one edge, we say that it is crossed in D. The following proposition is a trivial observation.

[1]Supported by Hunan Provincial Innovation Foundation For Postgraduate, China (No. CX2012B198; No.CX2012B195).

[2]Received September 2, 2012. Accepted December 20, 2012.

Proposition 1.1 *If there exists a clean edge e in a drawing D and contracting it results in a new drawing D^*, then $cr(D) \geq cr(D^*)$.*

Proposition 1.2 *If there exists a crossed edge e in a drawing D and contracting it results in a new drawing D^*, then $cr(D) \geq cr(D^*) + 1$.*

Proposition 1.3 *If G_1 is a subgraph of G_2, then $cr(G_1) \leq cr(G_2)$.*

Proposition 1.4 *Let G_1 be a graph homeomorphic to graph G_2, then $cr(G_1) = cr(G_2)$.*

Crossing number is an important parameter, which manifest the nonplanar of a given graph. It has not only theory significance but also great practical significance, early in the eighties of the 19th century. Bhatt and Leithon [1,2] showed that the crossing number of a network(graph) is closely related to the minimum layout area required for the implementation of a VLSI circuit for the network. Szekly [3] solved the very difficult problem of Edörs in dispersed geometry by crossing number of a graph. At present, the crossing number of a graph has widely used in VLSI layout, dispersed geometry, number theory, biological project and so on.

Calculating the crossing number of a given graph is NP-complete [4]. Only the crossing number of very few families of graphs are known exactly, some of which are the crossing number of generalized Petersen graph. Guy and Harary (1967) have shown that, for $k \geq 3$, the graph $P[2k,k]$ is homeomorphic to the Möbius ladder M_{2k}, so that its crossing number is one, and it is well known that $P[2k,2]$ is planar. Exoo and Harary, etc. researched on the crossing number of some generalized Petersen graph. In [5] they showed that

$$cr(P[n,2]) \leqslant \begin{cases} 0, & \text{for even}, \quad n \geq 4, \\ 3, & \text{for odd}, \quad n \geq 7. \end{cases}$$

and they also proved that $cr(P[3,2]) = 0$ and $cr(P[5,2]) = 2$. R.B.Richter and G.Salazar investigated the crossing number of the generalized Petersen graph P[N,3] and in [6] proved that(the graph P[9,3] is not be included)

$$cr(P[3k+h,3]) \leqslant \begin{cases} k+h, & \text{for k} \geq 3, \quad h \in \{0,2\}, \\ k+3, & \text{for k} \geq 3, \quad h = 1. \end{cases}$$

S.Fiorini and J.B.Gauci [7] study the crossing number of the generalized Petersen graph P[3k,k] and prover that $cr(P[3k,k]) = k$ for $k \geq 4$.

In this paper, we investigate the crossing number of the generalized Petersen graph $P[3k-1,k]$ and get the main result that

$$k \leq cr(P[3k-1,k]) \leq k+1 \text{ for } k \geq 3.$$

§2. Cartesian Products

Let S_3 denote the star-graph $K_{1,3}$ and P_n the path-graph with $n+1$ vertices, and consider the graph of the Cartesian product $S_3 \times P_n$, denoting the vertices $(0,i), (1,i), (2,i)$ and $(3,i)$ by

x_i, a_i, b_i and c_i, respectively for $(i = 0, 1, \cdots, n)$, where the vertices x_i represent the hubs of the star. In the drawing of $S_3 \times P_n$, we delete the path $\Gamma = (x_0, x_1, \cdots, x_n)$ which passes through the hubs of the stars. We let the subgraph of $((S_3 \times P_n) - \Gamma)$ induced by the vertices x_i, a_i, b_i and c_i be denoted by S^i. Also, the subgraph induced by the vertices $x_i, a_i, b_i, c_i, x_{i+1}, a_{i+1}, b_{i+1},$ and c_{i+1} is denoted by H^i, so that H^i is made up of S^i and S^{i+1} together with the three edges connecting the two stars, as illustrated in Figure 1.

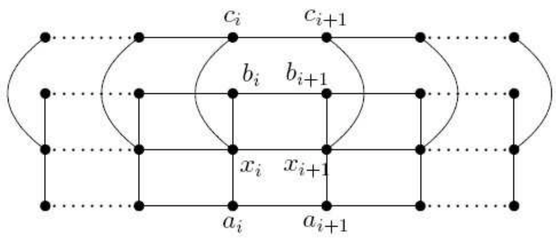


Figure 1 A good drawing of $S_3 \times P_n$

It is easy to obtain the following lemma 2.1 below, since the upper bound follows from the drawings of Figure 1, while the proof of the lower bound follows the same lines as that in Jendrol and Scerbova [8].

Lemma 2.1 *Let G denote the graph of Cartesian product $S_3 \times P_n (n \geq 1)$, with the path Γ joining the hubs of the stars deleted, that is, $G := ((S_3 \times P_n) - \Gamma)$. If D is a good drawing of G in which no star $S^i (i = 0, 1, \cdots, n)$ has a crossed edge, then $cr_D(G) = n - 1$.*

§3. The Generalized Petersen Graph $P[3k-1, k]$

The generalized Petersen graph $P[3k-1, k]$ of order $\{6k-2\}$ is made up of a principal cycle $C = \{u_1, u_2, \cdots, u_{3k-1}\}$, the spokes $u_i x_i$ and a adjoint principal cycle $\bar{C} = \{x_i, x_{k+i}, x_{2k+i}, \cdots, x_{3k-1}\}$, where $i = 1, 2, \cdots, 3k-1$ and addition is taken modulo $\{3k-1\}$. A drawing of $P[3k-1, k]$ is shown in Figure 2.

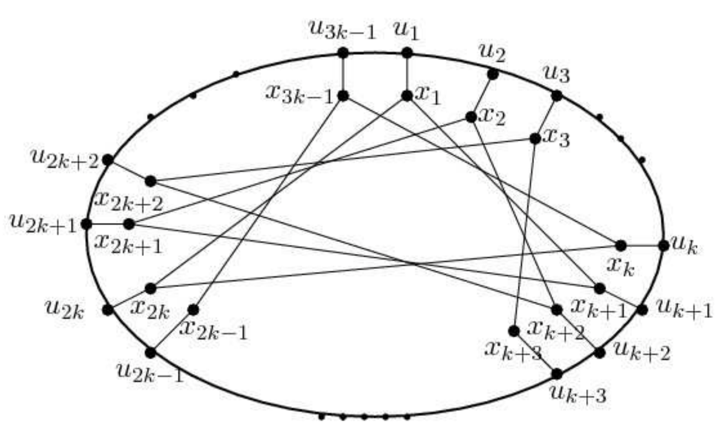


Figure 2

Let ϕ be a good drawing. In order to get an upper bound for the crossing number of $P[3k-1,k]$, we have shown a good drawing of $P[3k-1,k]$ in Figure 3.

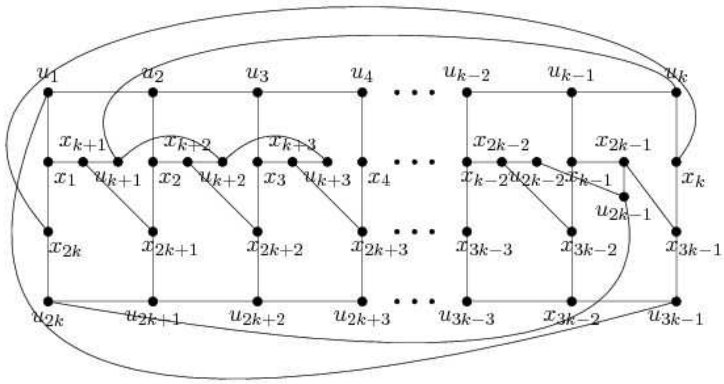


Figure 3: a good drawing of $P[3k-1,k]$

Figure 3 sets the upper bounded equal to $k+1$. To show that $cr(P[3k-1,k]) \geq k$, in the drawing of Figure 3, we note that by deleting the not crossed edges of $x_i x_{i+k}$ (where $k+1 \leq i \leq 2k-1$), and wipe away these 2-degree vertices $\{x_{k+1}, x_{k+2}, \cdots, x_{2k-1}, x_{2k+1}, \cdots, x_{3k-1}\}$. Considering the spoke $x_{2k}u_{2k}$ is clean or crossed, we now consider the following two cases.

Case 1 First, we consider that the spoke $x_{2k}u_{2k}$ is clean. Then contract the spoke $x_{2k}u_{2k}$ to the vertex u_{2k}, we obtain the graph G_k is shown in Figure 4, such that $G_k \supseteq (S_3 \times P_{k-1} - \Gamma)$. Obviously, $P[3k-1,k]$ contains G_k as a subgraph. By Proposition 1.1 and Proposition 1.3, we have $cr(P[3k-1,k]) \geq cr(G_k)$. Thus, in order to get a lower bound for the crossing number of $P[3k-1,k]$, we can simply consider the crossing number of the graph G_k as shown in Figure 4.

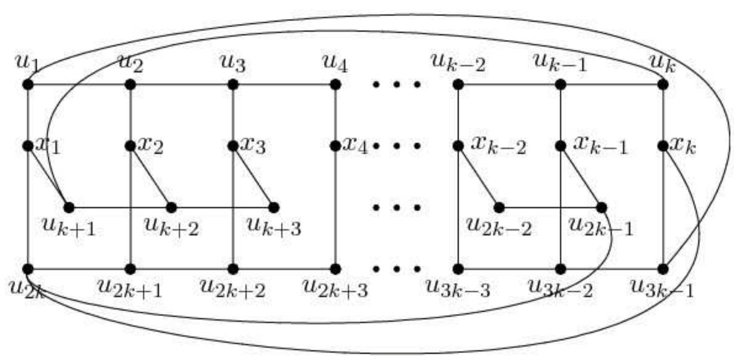


Figure 4: A good drawing of G_k

Lemma 3.1 $cr(P[3k-1,k]) \geq k \quad (k \geq 3)$.

Proof For $k=1$, $P[2,1]$ is a planar graph. For $k=2$, from above $cr(P[5,2]) = 2$. Now we consider the case for $k \geq 3$.

For $k=3$, from [6] we have $cr(P[8,3]) = 4 \geq 3$. The theorem is true for $k=3$. Now suppose that for $k > 3$. In order to prove that $cr(P[3k-1,k]) \geq k$, we only should to prove

that $cr(G_k) \geq k$ for $k > 3$. We assume that t is the least value of the crossing number k for which $cr(G_t) \leq t-1$, t is greater than 3. We also note that the deletion of the vertex x_i and the edges incident to it (for values of i between 2 and $k-1$) from G_k yields a graph homeomorph to G_{k-1}. Therefore, since G_t contains G_{t-1} as a subgraph, we have $cr(G_t) \geq cr(G_{t-1}) \geq t-1$ by minimality of t. Thus, we only need to show that $cr(G_t) \neq t-1$. By assuming, for contradiction that it is

$$cr(G_t) = t - 1. \qquad (1).$$

We divide the problem into three cases to prove that $cr(G_t) \geq t$.

Case 1 The spoke $x_{2k}u_{2k}$ is not crossed.

Case 1.1 First we consider an optimal drawing D of G_t and assume that a star edge of $\{x_iu_i, x_iu_{i+t}, x_iu_{i+2t-1}\}$ for $2 \leq i \leq t-1$, makes a positive contribution to the crossing number of G_t. In this case, when we delete the hub $x_i(2 \leq i \leq t-1)$ we get an induced drawing D_1 of homeomorph of G_{t-1} such that

$$\begin{aligned} t-1 &= cr(G_t) \geq cr_{D_1}(G_{t-1}) + 1 \\ &\geq (t-1) + 1, \end{aligned}$$

by the inductive hypothesis, a contradiction.

Case 1.2 Now we consider an optimal drawing D of G_t and assume that the star edge of $\{x_1u_1, x_1u_{k+1}, x_1u_{2k}\}$ or $\{x_ku_k, x_ku_{3k-1}, x_ku_{2k}\}$ makes a positive contribution to the crossing number of G_t. In this case, we delete the hub $x_i(x = 1$ or $k)$ because there is no crossing in the $\{3t-1\}$ principal cycle C. So the edge $\{u_{2k}u_{2k+1}\}$ or $\{u_1u_{3k-1}\}$ is clean. Then contract the edge $\{u_{2k}u_{2k+1}\}$ or $\{u_1u_{3k-1}\}$. Following the same arguments presented in Case 1.1, we get an induced drawing D_2 of homeomorph of G_{t-1} such that

$$\begin{aligned} t-1 &= cr(G_t) \geq cr_{D_2}(G_{t-1}) + 1 \\ &\geq (t-1) + 1, \end{aligned}$$

by the inductive hypothesis, a contradiction.

Case 1.3 Thus, we can assume that all the $\{t-1\}$ crossings of G_t are self-intersections of the $\{3t-1\}$ principal cycle C made up of the edges u_iu_{i+1} for $1 \leq i \leq 3t-1$ addition modulo $\{3t-1\}$.

Therefore, there exists an optimal drawing D_3 of G_t such that in D_3 the edges of the stars do not contribute to the crossing number. We divide the problem in to the following different subcases.

Subcase 1.3.1 If there is an edge e in u_iu_{i+1} in D_3 which is crossed twice and more, then deleting e together with the two other edges at distance t from e along C. As it is the edges $\{u_iu_{i+1}, u_{k+i}u_{k+i+1}, u_{2k+i-1}u_{2k+i}\}$, then we get a subgraph homeomorphic to $(S_3 \times P_{t-1} - \Gamma)$. Therefore, by Lemma 2.1 above.

$$\begin{aligned} t-1 &= cr(G_t) \geq cr(S_3 \times P_{t-1} - \Gamma) + 2 \\ &= (t-2) + 2 = t, \text{ a contradition.} \end{aligned}$$

Subcase 1.3.2 If there are two edges in D_3 at distance t or $t-1$ from each other which are crossed but do not cross each other, then repeating the same procedure as in case 1. So we have

$$\begin{aligned} t-1 &= cr(G_k) \geq cr(S_3 \times P_{t-1} - \Gamma) + 2 \\ &= (t-2) + 2 = t, \text{ acontradiction.} \end{aligned}$$

We can therefore assume hereafter that in D_3 there is no edge which is crossed twice and no two edges at distant t from each other giving a contribution of two to the crossing. As it is, now we consider the case there is an edge e in D_3 which is crossed once. There remains to show that if in D_3:

(i) There are no two edges at a distance t which not intersected, or

(ii) There are two edges at a distance t from each other which are pairwise intersecting, then in both cases we get a contradiction.

Let us first assume that no two edges at a distance t or $t-1$ from each other can be found such that they are both intersected. We divide the $\{3t-1\}$ principal cycle C into two $\{t\}$-sectors and one $\{t-1\}$-sector such that the number of crossed edges in each sector is p, q and r, respectively. Since in Sector 1 there are p crossed edges which cannot be matched to crossed edges in Sector 2 and 3. Hence in each of Sector 2 and 3 there are p edges which are not intersected. Similarly for q and r. Thus, the number of uncrossed edges is at least $2(p+q+r)$. However, the total number of edges$\geq$(numbers of crossed edges)+(number of uncrossed edges) is,

$$\begin{aligned} 3t-1 &\geq (p+q+r) + 2(p+q+r) \\ &= 3(p+q+r). \end{aligned}$$

Let $p+q+r = x$. This implies that $3t-3x \geq 1$. As t is the least value of k for the crossing number, then $t \leq x$. Thus $3t-3x \geq 1$, a contradiction.

We now assume that there are two edges in D_3 at distant t or $t-1$ from each other that at intersect each other. That is, if $u_iu_{i+1}(1 \leq i \leq t$ addition taken modulo t) is intersected by an edge e, then $e \in \{u_{i+t}u_{i+t+1}, u_{i+2t-1}u_{i+2t}\}$. Without loss of generality, we assume that $e = \{u_{i+t}u_{i+t+1}\}$ and consequently, that the edge $u_{i+2t-1}u_{i+2t}$ is not intersected.

We consider the subgraph H induced by $S(x_i) \cup S(x_{i+1}) \setminus \{x_iu_{i+t}, x_{i+1}u_{i+t+1}\}$ (shown in Figure 5(a)). This is a 6-circuit none of whose edges is intersected, with the sole exception of u_iu_{i+1} which is intersect once by $u_{i+t}u_{i+t+1}$. ,Thus, H is planarly embedded and without loss of generality, we let $u_{i+1} \in Int(H)$ and $u_{i+t+1} \in Ext(H)$(since if these vertices are both in Int(H) or in Ext(H), then the edges u_iu_{i+1} is crossed an even number of times). Therefore, we have the subgraph shown in the drawing of Figure5(b). Now x_{i+2} either lies in $Int(H)$ or $Ext(H)$ and none of the edges of S_{i+2} can be crossed. Also, the edges of the subgraph in Figure 4(b) cannot be crossed(apart from the crossing shown), giving us the required contradiction. Therefore, $cr(G_k) \neq t-1$. So, formula (1) does not hold. So, we have shown that $cr(G_k) \geq k$, hence

$$cr(P[3k-1,k]) \geq cr(G_k) \geq k \quad (k \geq 3). \tag{2}$$

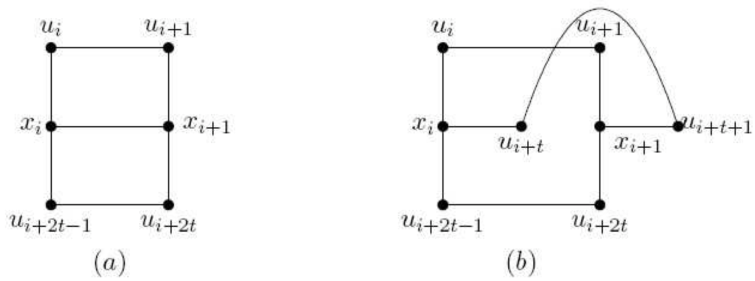


Figure 5

Case 2 Now we consider that the spoke $x_{2k}u_{2k}$ is crossed. Using the analogous arguments presented in Case 1. Then contract the spoke $x_{2k}u_{2k}$ to the vertex u_{2k}. We obtain the graph G_k is same shown in Figure 4, such that $G_k \supseteq (S_3 \times P_{k-1} - \Gamma)$. By Proposition 1.2 and Proposition 1.3, then we have $cr(P[3k-1,k]) \geq cr(G_k)+1$. We can follow the same conclusion $cr(G_k) \geq k$ presented in Case 1. Hence

$$cr(P[3k-1,k]) \geq k+1. \tag{3}$$

As a result, from all the above cases, combine with formula (2) and (3), we have shown that $cr(P[3k-1,k]) \geq k$. □

Theorem 3.1 $k \leq cr(P[3k-1,k]) \leq k+1 \quad (k \geq 3)$.

Proof A good drawing of $P[3k-1,k]$ in Fig.3 shows that $cr(P[3k-1,k]) \leq k+1$ for $k \geq 3$. This together with Lemma 3.1 immediately indicate that

$$k \leq cr(P[3k-1,k] \leq k+1 \quad (k \geq 3).$$ □

We end this paper by presenting the following conjecture.

Conjecture $cr(P[3k-1,k]) = k+1 \quad (k \geq 3)$.

References

[1] S.N.Bhatt, F.T.Leighton, A framework for solving VLSI graph layout problems, *J.Comput.System Sci.*, 1984, 28: 300-343.

[2] F.T.Leighton, New lower bound techniques for VLSI, *Math.System Theory*, 1984, 17: 47-70.

[3] L.A.Szekely, Crossing numbers and hard Erdös problems in discrete geometry, *Combinatorics, Probability and Computing*, 1997, 6: 353-358.

[4] M.R.Garey, D.S.Johnson, Crossing number is NP-complete, *SIAM J Algebric Discrete Mathematics*, 4(1993), 312-316.

[5] Jager P. de, J.W.Ren, Phase portraits for quadratic systems with a higher order singularity, *Proceedings ICIAM*, 87(1987), 75-87.

[6] R.B.Richter, G.Salazar, The crossing number of $P(N,3)$, *Graphs and Combinatorics*, 18(2002), 381-394.

[7] S.Fiorini, J.B.Gauci, The crossing numbers of the generalized Petersen graph $P[3k,k]$, *Mathematica Bohemica*, 128(2003), 337-347.

[8] S.Jendrol, M.Scerbova, On the crossing numbers of $S_m \times C_n$, *Cas.Pest.Mat.*, 107(1982), 225-230.

We know nothing of what will happen in future, but by the analogy of past experience.

By Abraham Lincoln, a American president.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in **Mathematical Combinatorics (International Book Series** (*ISBN 978-1-59973-211-4*). An effort is made to publish a paper duly recommended by a referee within a period of 3 months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

December 2012

Contents

ISBN 9781599732114

90000 >

9 781599 732114